Financial Engineering and Computations

Basic Financial Mathematics

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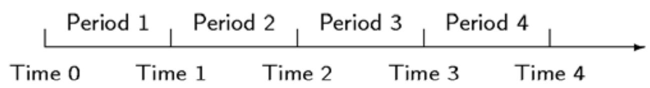
此章內容

- Financial Engineering & Computation教課書的第三章　Basic Financial Mathematics
- C++財務程式設計的第三章（3-4,3-5）

Outline

- Time Value of Money
- Annuities
- Amortization
- Yields
- Bonds

Time Value of Money

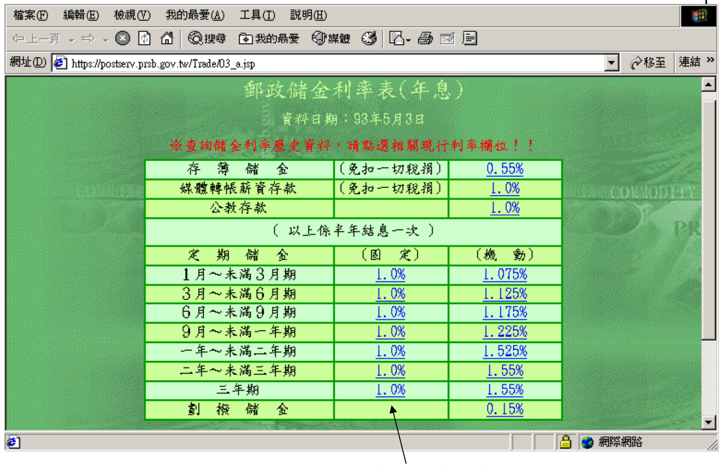


$$PV = FV(1+r)^{-n}$$

$$FV = PV(1+r)^{n}$$

- FV: future value
- PV: present value
- r: interest rate
- n: period terms

Quotes on Interest Rates



郵政儲金利率表(年息)

資料日期：93年5月3日

※查詢儲金利率歷史資料，請點選相關現行利率欄位！！

存 簿 儲 金	(免扣一切稅捐)	0.55%
媒體轉帳薪資存款	(免扣一切稅捐)	1.0%
公教存款		1.0%
(以上係半年結息一次)		
定 期 儲 金	(固 定)	(機 動)
1月～未滿3月期	1.0%	1.075%
3月～未滿6月期	1.0%	1.125%
6月～未滿9月期	1.0%	1.175%
9月～未滿一年期	1.0%	1.225%
一年～未滿二年期	1.0%	1.525%
二年～未滿三年期	1.0%	1.55%
三年期	1.0%	1.55%
劃 撥 儲 金		0.15%

Annualized rate.

r is assumed to be constant in this lecture.

Time Value of Money

- Periodic compounding

  (If interest is compounded *m* times per annum)

$$FV = PV\left(1+\frac{r}{m}\right)^{nm} \qquad (3.1)$$

- Continuous compounding

$$FV = PVe^{rn}$$

$$\lim_{t\to\infty}(1+\frac{1}{t})^t = e \rightarrow \lim_{m\to\infty}(1+\frac{r}{m})^{nm} = \lim_{m\to\infty}(1+\frac{1}{m/r})^{\frac{m}{r}rn} = e^{rn}$$

- Simple compounding

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
  - Bond equivalent yield (BEY)

    --Annualize yield with semiannual compounding
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
  - Mortgage equivalent yield (MEY)

    --Annualize yield with monthly compounding
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$

Equivalent Rate per Annum

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be 1.1025 one year from now.

$$(1+(0.1/2))^2 = 1.1025$$

- The rate is equivalent to an interest rate of 10.25% compounded once *per annum*.

Conversion between compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent compounded m times per annum.
- Then $\left(1+\frac{r_2}{m}\right)^m = e^{r_1}$
- Therefore $r_1 = m\ln\left(1+\frac{r_2}{m}\right) \Rightarrow r_2 = m\left(e^{\frac{r_1}{m}}-1\right)$

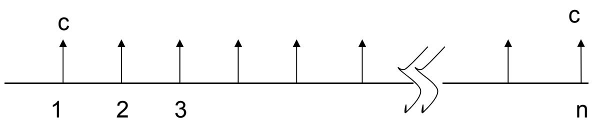
Are They Really "Equivalent"?

- Recall r_1 and r_2 on the previous example.
- They are based on different cash flow.
- In what sense are they equivalent?

Annuities

- An annuity pays out the same C dollars at the end of each year for years.
- With a rate or r , the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\frac{(1+r)^n-1}{r} \qquad (3.4)$$



General Annuities

- If m payments of C dollars each are received per year, then Eq.(3.4) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{n\,m}-1}{\frac{r}{m}}$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\frac{1-\left(1+\frac{r}{m}\right)^{-n\,m}}{\frac{r}{m}} \qquad (3.6)$$

Perpetual annuity

- An annuity that lasts forever is called a perpetual annuity. We can drive its *PV* from Eq.(3.6) by letting ***n*** go to infinity:

$$PV = \lim_{n\to\infty} \sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = \lim_{n\to\infty} C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n\,m}}{\frac{r}{m}} = \frac{mC}{r}$$

- This formula is useful for valuing *perpetual fix-coupon debts.*

Amortization

- It is a method of repaying a loan through regular payment of interest and principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the the interest part of the payment diminishes.

See next example!

Example: Home mortgages

- Consider a 15-year, \$250,000 loan at 8.0% interest rate, repay the loan per month.
- Because *PV = 250,000*, *n = 15*, *m= 12*, and *r = 0.08* we can get a monthly payment *C is \$2,389.13.*

$$\$250000 = \frac{C}{(1+\frac{0.08}{12})} + \frac{C}{(1+\frac{0.08}{12})^2} + \ldots\ldots + \frac{C}{(1+\frac{0.08}{12})^{12\times 15}}$$

$$= \sum_{i=1}^{180} C\left(1+\frac{0.08}{12}\right)^{-i} = C\left(\frac{1-(1+\frac{0.08}{12})^{-180}}{0.08/12}\right) \Rightarrow C = 2389.13$$

249277.536×(0.08/12)

Payment − Interest

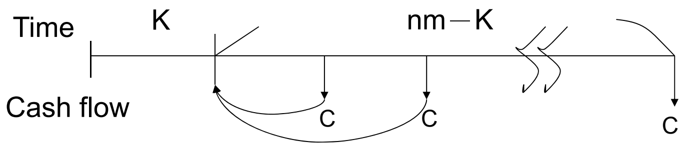
Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

We compute it in last page

Calculating the Remaining Principal

- Right after the *k*th payment, the remaining principal is the PV of the future *nm-k* cash flows,

$$C(1+\frac{r}{m})^{-1}+C(1+\frac{r}{m})^{-2}+.....+C(1+\frac{r}{m})^{-(nm-k)}=C\frac{1-(1+\frac{r}{m})^{-nm+k}}{\frac{r}{m}}$$



Yields

- The term **yield** denotes the return of investment.
- It has many variants.
  - (1) Nominal yield (coupon rate of the bond)
  - (2) Current yield
  - (3) Discount yield
  - (4) CD-equivalent yield

Discount Yield

- U.S *Treasury bills* is said to be issue on a discount basis and is called a discount security.
- When the discount yield is calculated for short-term securities, a year is assumed to have **360 days**.
- The discount yield (discount rate) is defined as

Interest

$$\frac{par\ \ value-purchase\ \ price}{par\ \ value}\times\frac{360\ days}{number\ of\ \ days\ to\ maturity} \qquad (3.9)$$

Interest rate

Annualize

CD-equivalent yield

- It also called the money-market-equivalent yield.
- It is a simple annualized interest rate defined as

$$\frac{\text{par value}-\text{purchase price}}{\text{purchase price}}\times\frac{\textbf{365}\text{ days}}{\text{number of days to maturity}} \qquad (3.10)$$

Example 3.4.1: Discount yield

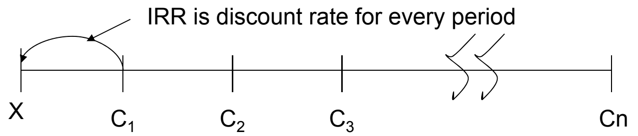
- If an investor buys a U.S. $ 10,000, 6-month T-bill for U.S. $ 9521.45 with 182 days remaining to maturity.
- $Discount\,yield = \frac{10000 - 9521.45}{10000} \times \frac{360}{182} = 0.0947$
- *Show that discount yield<CD equivalent yield*

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment's PV with its price *X*.

$$X = C_1 \times (1+IRR)^{-1} + C_2 \times (1+IRR)^{-2} + ... + C_n \times (1+IRR)^{-n}$$

- IRR assumes all cash flows are reinvested at the same rate as the internal rate of return.
- It doesn't consider the reinvestment risk.



Evaluating real investment with IRR

- Multiple IRR arise when there is more than one sign reversal in the cash flow pattern, and it is also possible to have no IRR.
- Evaluating real investment, IRR rule breaks down when there are multiple IRR or no IRR.
- Additional problems exist when the term structure of interest rates is not flat.

  →there is ambiguity about what the appropriate hurdle rate (cost of capital) should be.

Class Exercise

- Assume that a project has cash flow as follow respectively, and initial cost is $1000 at date 0, please calculate the IRR. If cost of capital is 10%, do you think it is a good project?

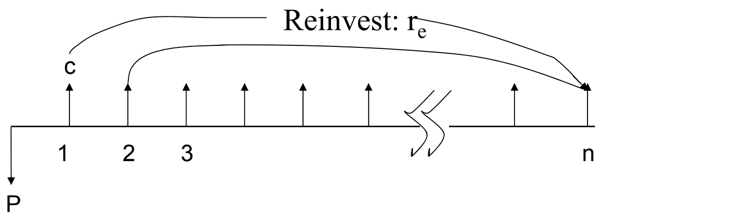
CF at date					
0	1	2	3	4	IRR
-1000	800	1000	1300	-2200	?

Class Exercise (Excel)

	Time	CF	
12	Time	CF	
13	0	-1000	
14	1	800	
15	2	1000	
16	3	1300	
17	4	-2200	=IRR(B13:B17,0.1)
18		7%	
19		37%	=IRR(B13:B17,0.2)
20			
21			

Multiple IRR

Holding Period Return



- The FV of investment in n period is $FV = P(1+y)^n$
- Let the reinvestment rates r_e, the FV of per cash income is

$$C\times(1+r_e)^{n-1} + C\times(1+r_e)^{n-2} + ... + C\times(1+r_e) \ + C \longrightarrow \text{Value is given}$$

- We define HPR (y) is

$$P(1+y)^n = C\times(1+r_e)^{n-1} + C\times(1+r_e)^{n-2} + ... + C\times(1+r_e) \ + C$$

Methodology for the HPR(y)

- Calculate the FV and then find the yield that equates it with the P
- Suppose the reinvestment rates has been determined to be r_e.

Step	Periodic compounding	Continuous compounding
(1)Calculate the future value	$FV = \sum_{t=1}^{n} C\ (1+r_e)^{n-t}$	$FV = C\times\frac{(e^{r_e n}-1)}{e^{r_e}-1}$
(2)Find the HPR	$y = \sqrt[n]{\frac{FV}{P}} - 1$	$y = \frac{1}{n}\ln(\frac{FV}{P})$

Example 3.4.5:HPR

- A financial instrument promises to pay \$ 1,000 for the next 3 years and sell for \$ 2,500. If each cash can be put into a bank account that pays an effective rate of 5%.
- The FV is $\sum_{t=1}^{3} 1000\times(1+0.05)^{3-t} = 3152.5$
- The HPR is $2500(1+HPR)^3 = 3125.5$

$$\Rightarrow HPR = \left(\frac{3152.5}{2500}\right)^{1/3} - 1 = 0.0804$$

Numerical Methods for Yield

- Solve $f(r)=\sum_{t=1}^{n}\frac{C_t}{(1+r)^t}-x=0,\ for\ r\geq -1,\ x\ is\ market\ price$

  $Recall\ \ X=C_1\times(1+IRR)^{-1}+...+C_n\times(1+IRR)^{-n}$

  $\Rightarrow C_1\times(1+IRR)^{-1}+...+C_n\times(1+IRR)^{-n}-X=0$

  $Let\ \ f(r)=C_1\times(1+r)^{-1}+....C_n\times(1+r)^{-n}-X$

- The function *f(r)* is monotonic in *r*, if $C_t > 0$ for all t, hence a unique solution exists.

The Bisection Method

- Start with *a* and *b* where *a* < *b* and *f*(*a*) *f*(*b*) < 0.
- Then *f(r)* must be zero for some *r*∈(*a* , *b*).
- If we evaluate *f* at the midpoint *c*≡ (*a* + *b*) / 2

  (1) *f(a) f(c)* < *0* → *a<r<c*

  (2) *f(c) f(b)* < *0* → *c<r<b*

- After *n* steps, we will have confined *r* within a bracket of length $(b - a) / 2^n$.

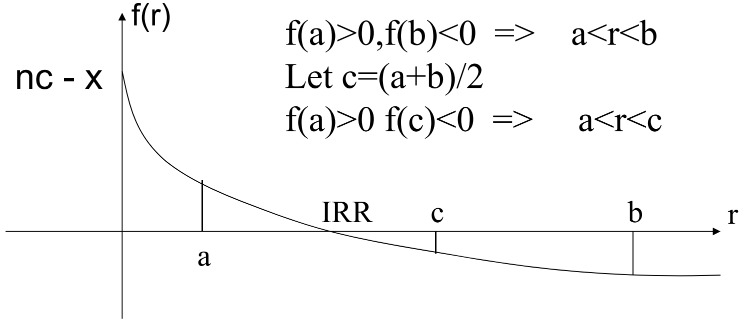
Bisection Method

- Let $f(r)=C\times(1+r)^{-1}+C\times(1+r)^{-2}+...+C\times(1+r)^{-n}-X$
- Solve *f(r)=0*

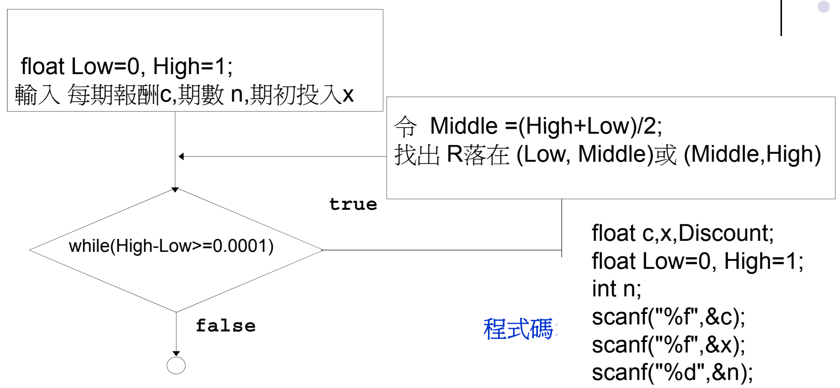
f(a)>0,f(b)<0 => a<r<b

Let c=(a+b)/2

f(a)>0 f(c)<0 => a<r<c



C++:使用while 建構二分法



程式碼:

```
float c,x,Discount;
float Low=0, High=1;
int n;
scanf("%f",&c);
scanf("%f",&x);
scanf("%d",&n);
while(High-Low>=0.0001)
{

}
printf("Yield rate=%f",High);
```

用Bisection method縮小根的範圍

- 已知$f(r)=c\times(1+r)^{-1}+c\times(1+r)^{-2}+...+c\times(1+r)^{-n}-x$
  - f(r)<0 → r >R
  - f(r)>0 → r<R
- 令 Middle=(High+Low)/2
  - 將根的範圍從(Low,High)縮減到
    - (Low,Middle)
    - (Middle,High)

$c\times(1+r)^{-1}+c\times(1+r)^{-2}+...+c\times(1+r)^{-n}$

用計算債券的公式計算

縮小根的範圍

```
float  Middle=(Low+High)/2;
float Value=0;
for(int i=1;i<=n;i=i+1)
{
 Discount=1;
 for(int j=1;j<=i;j++)
 {
          Discount=Discount/(1+Middle)
 }
  Value=Value+Discount*c;
}
Value=Value-x;
if(Value>0)
          { Low=Middle;}
  else
          {High=Middle;}
```

計算 IRR (完整程式碼)

用while控制根的範圍

計算 $c\times(1+r)^{-1}+c\times(1+r)^{-2}+...+c\times(1+r)^{-n}$

計算 $(1+r)^{-i}$

縮小根的範圍

```
float c,x,Discount;
float Low=0, High=1;
int n;
scanf("%f",&c);
scanf("%f",&x);
scanf("%d",&n);
while(High-Low>=0.0001)
{
float  Middle=(Low+High)/2;
float Value=0;
for(int i=1;i<=n;i=i+1)
{
 Discount=1;
 for(int j=1;j<=i;j++)
 {
          Discount=Discount/(1+Middle);
 }
  Value=Value+Discount*c;
}
Value=Value-x;
if(Value>0)
          { Low=Middle;}
  else
          {High=Middle;}
}
printf("Yield rate=%f",High);
```

Homework 1

- 第三章第十題

　假定有一個投資計畫，該投資計畫可在現在獲得9702元收益，在第一期結束時需支付19700元，第二期計畫結束時，可再獲得10000元，請仿照上述求內部收益率的程式，撰寫程式使用二分法求內部收益率，請問這種解法會不會碰到問題？請用Newton method 驗證計算結果

C++財務程式設計

The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation X_0 to a root of $f(x)=0$.
- Then

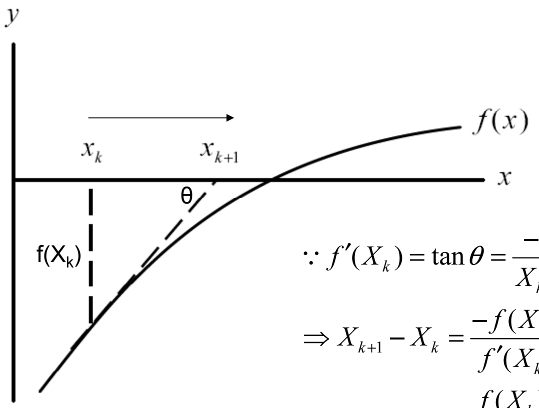
$$x_{k+1}\equiv x_k-\frac{f(x_k)}{f'(x_k)} \qquad (3.15)$$

- When computing yields,

$$f'(x)=-\sum_{t=1}^{n}\frac{t\,C_t}{(1+x)^{t+1}}$$

※ **Recall the bisection method, the X here is r (yield) in the bisection method!**

Figure3.5: Newton-Raphson method



$$\because f'(X_k) = \tan\theta = \frac{-f(X_k)}{X_{k+1} - X_k}$$

$$\Rightarrow X_{k+1} - X_k = \frac{-f(X_k)}{f'(X_k)}$$

$$\Rightarrow X_{k+1} = X_k - \frac{f(X_k)}{f'(X_k)}$$

If $f(X_{k+1})=0$, we can obtain X_{k+1} is yield

Computed by Excel

- **Yield的計算**
  - **RATE(nper, pmt, pv, fv, type)**。
  - Nper：年金的總付款期數。
  - Pmt ：各期所應給付 (或所能取得) 的固定金額。
  - Pv ：期初應給付或取得的金額
  - Fv ：最後一次付款完成後，所應付出或獲得的現金餘額 。
  - Type 0=>期末支付 1=>期初支付

Example

	A	B	C	D	E	F
1	某政府公債票面利率為5%，發行價格為$95，票面價格為$100，半年支付一次，到期期間為10年，求YTM? YTM=2.83%*2=5.66%					
2	Nper	20				
3	Pmt	2.5				
4	Pv	-95				
5	Fv	100				
6	Type	0				
7						
8	YTM	2.83%				
9						

=RATE(B2,B3,B4,B,5B6)

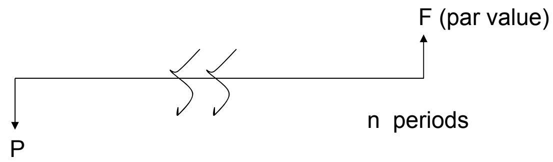
Bond

- A bond is a contract between the issuer (borrower) and the bondholder (lender).
- Bonds usually refer to long-term debts.
- Callable bond, convertible bond.
- Pure discount bonds vs. level-coupon bond

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays F dollars in n periods is $P = \frac{F}{(1+r)^n}$

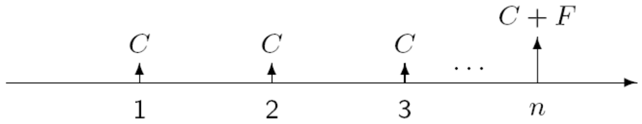
  where r is the interest rate per period
- No coupon is paid before bond mature.
- Can meet future obligations without reinvestment risk.



Level-Coupon Bonds

- It pays interest based on coupon rate and the par value, which is paid at maturity.
- ***F*** denotes the par value and ***C*** denotes the coupon.

$$P = C\times(1+r)^{-1} + C\times(1+r)^{-2} + \ldots + C\times(1+r)^{-n} + F\times(1+r)^{-n}$$



Pricing of Level-Coupon Bonds

$$P = \frac{C}{(1+\frac{r}{m})} + \frac{C}{(1+\frac{r}{m})^2} + \ldots\ldots + \frac{C}{(1+\frac{r}{m})^{nm}} + \frac{F}{(1+\frac{r}{m})^{nm}}$$

$$= \sum_{i=1}^{nm} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^{nm}} = C\left(\frac{1-(1+\frac{r}{m})^{-nm}}{\frac{r}{m}}\right) + \frac{F}{(1+\frac{r}{m})^{nm}} \qquad (3.18)$$

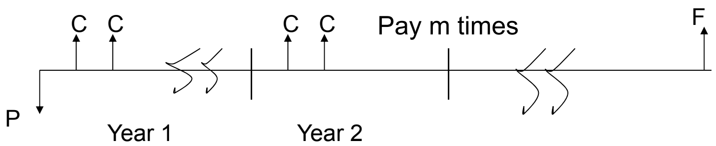
where

n: time to maturity (in years)

m : number of payments per year.

r : annual rate compounded m times per annum.

$C = Fc/m$ where c is the annual coupon rate.



Yield To Maturity

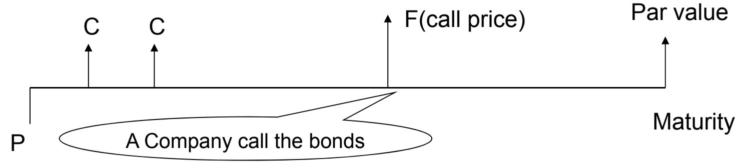
- The YTM of a level-coupon bond is its IRR when the bond is held to maturity.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$P = \frac{5}{(1+\frac{0.15}{2})} + \ldots\ldots \frac{5}{(1+\frac{0.15}{2})^{20}} + \frac{100}{(1+\frac{0.15}{2})^{20}}$$

$$= 5\times\frac{1-(1+(0.15/2))^{-2\times 10}}{0.15/2} + \frac{100}{(1+(0.15/2))^{2\times 10}} = 74.5138$$

Yield To Call

- For a callable bond, the yield to states maturity measures its yield to maturity as if were not callable.
- The yield to call is the yield to maturity satisfied by Eq(3.18), when ***n*** denoting the number of remaining coupon payments until the first call date and ***F*** replaced with call price.



Price Behaviors

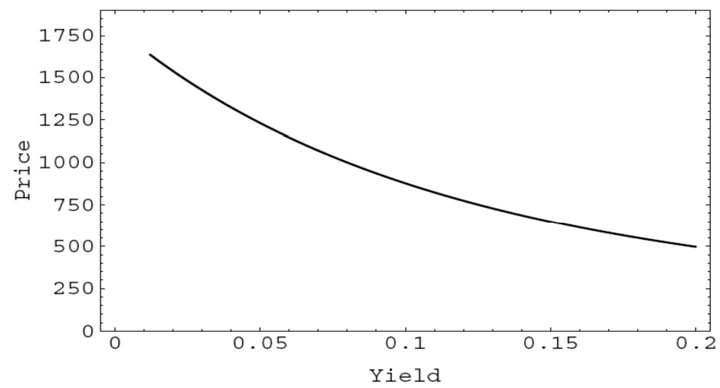
- Bond price falls as the interest rate increases, and vice versa.
- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate.
  - at par (at its par value) when its coupon rate is equal to the market interest rate.
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

Figure 3.8: Price/yield relations

Yield (%)	Price (% of par)	
7.5	113.37	→Premium bond
8.0	108.65	
8.5	104.19	
9.0	100.00	→Par bond
9.5	96.04	→Discount bond
10.0	92.31	
10.5	88.79	

Figure 3.9: Price vs. yield.

Plotted is a bond that pays 8% interest on a par value of $1,000,compounding annually. The term is 10 years.



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a year.
- Example: For coupon-bearing Treasury securities, the number of days between June 17, 1992, and October 1, 1992, is *106*.

→13 days (June), 31 days (July), 31 days (August), 30 days (September), and 1 day (October).

Day Count Conventions:30/360

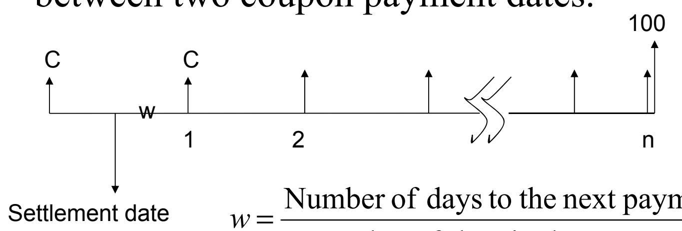
- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is *104*.
  - 13 days (June), 30 days (July), 30 days (August), 30 days (September), and 1 day (October).
- In general, the number of days from date1 to date2 is

$$360 \times (y2 - y1) + 30 \times (m2 - m1) + (d2 - d1)$$

Where Date1 ≡ (y1,m1, d1) Date ≡ (y2,m2, d2)

Bond price between two coupon date (Full Price, Dirty Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.



$$w = \frac{\text{Number of days to the next payment date}}{\text{Number of days in the coupon period}}$$

$$\text{DirtyPrice} = C\times(1+r)^{-\omega} + C\times(1+r)^{-\omega-1} + \dots\dots + C\times(1+r)^{-\omega-n+1} + 100\times(1+r)^{-\omega-n+1}$$

Accrued Interest

- The original bond holder has to share accrued interest in 1-ω period
  - Accrued interest is $C\times(1-\omega)$
- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the *clean price.*
- Dirty price= Clean price+ Accrued interest

Example 3.5.3

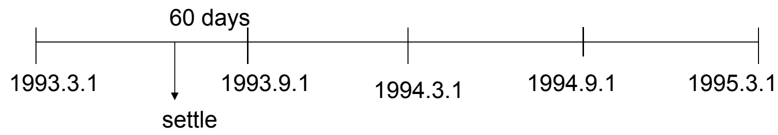
- Consider a bond with a 10% coupon rate, par value$100 and paying interest semiannually, with clean price 111.2891. The maturity date is March 1, 1995, and the settlement date is July 1, 1993. The yield to maturity is 3%.

Example: solutions

- There are ***60*** days between July 1, 1993, and the next coupon date, September 1, 1993.
- The ω= 60/180, C=5,and accrued interest is

  5× (1-(60/180)) =3.3333
- Dirty price=114.6224

  clean price = 111.2891



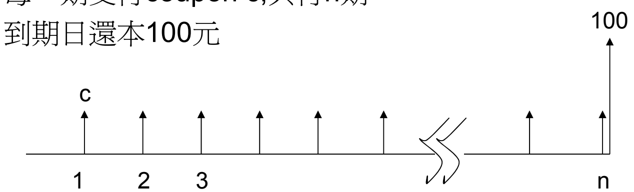
Exercise 3.5.6

- Before: A bond selling at par if the yield to maturity equals the coupon rate. (But it assumed that the settlement date is on a coupon payment date).
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.

  →The short reason: Exponential growth is replaced by linear growth, hence "overpaying" the coupon.

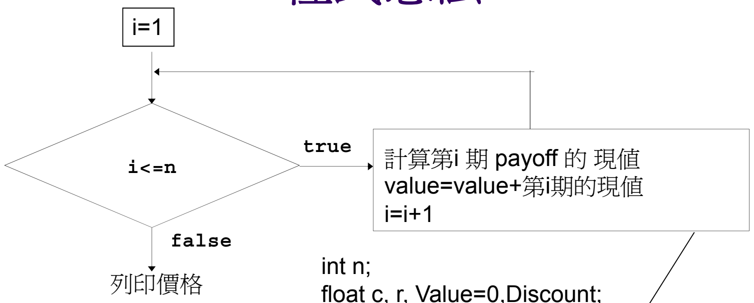
C++:計算債券價格

- 考慮債券價格的計算
  - 假定單期利率為r
  - 每一期支付coupon c,共付n期
  - 到期日還本100元



債券價格 $P = c\times(1+r)^{-1} + c\times(1+r)^{-2} + ... + c\times(1+r)^{-n} + 100\times(1+r)^{-n}$

程式想法




```
int n;
float c, r, Value=0,Discount;
scanf("%d",&n);
scanf("%f",&c);
scanf("%f",&r);
for(int i=1;i<=n;i=i+1)
{

}
printf("BondValue=%f",Value);
```

計算第i 次 payoff的 現值

- i<n 現值= $(1+r)^{-i}\times c$
- i=n 現值= $(1+r)^{-n}\times(c+100)$
- 用for計算 $(1+r)^{-i}$

計算第i 次 payoff的 現值

計算 $(1+r)^{-i}$

考慮最後一期本金折現

```
Discount=1;
for(int j=1;j<=i;j++)
{
 Discount=Discount/(1+r);
}
Value=Value+Discount*c;
if(i==n)
{
 Value=Value+Discount*100;
}
```

完整程式碼(包含巢狀結構)

```
#include <stdio.h>
void main()
{
        int n;
        float c, r, Value=0,Discount;
        scanf("%d",&n);
        scanf("%f",&c);
        scanf("%f",&r);
        for(int i=1;i<=n;i=i+1)
        {
        Discount=1;
        for(int j=1;j<=i;j++)
        {
         Discount=Discount/(1+r);
        }
        Value=Value+Discount*c;
        if(i==n)
        {
         Value=Value+Discount*100;
        }
        }
  printf("BondValue=%f",Value);
}
```

第i 次 payoff的 現值

Value爲前 i次payoff 現值

Use pow(x,y) to evaluate x^y

Homework 2

- Program exercise:

  Calculate the dirty and the clean price for a bond under actual/actual and 30/360 day count conversion.

  Input: Bond maturity date, settlement date, bond yield, and the coupon rate.

  The bond is assumed to pay coupons semiannually.