Bond Price Volatility

Financial Engineering and Computations

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此章內容

- Financial Engineering & Computation教課書的第四章　Bond Price Volatility
- C++財務程式設計的第三章（3-4,3-5）

Outline

- Price Volatility
- Duration
- Convexity
- Immunization

Price Volatility

- Price volatility measures the sensitivity of the percentage price change to changes in interest rates (interest rate risk).
- It is key to the risk management of interest-rate-sensitive securities.
- Define price volatility by

$$-\frac{\partial P/P}{\partial y} \longrightarrow \text{It is also so-call modified duration!}$$

$$\frac{\partial P}{P}(percent\ \ price\ change) \approx -D \times \partial y$$

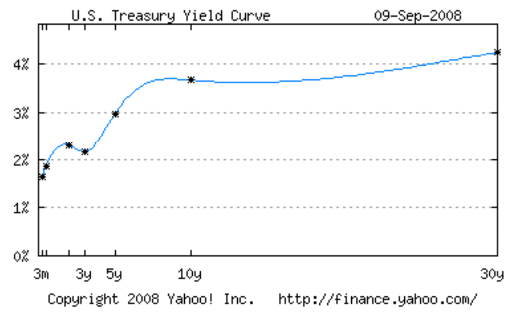
Numerical Example: Percentage Change of Bond Price

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1 %, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

General speaking, the duration we talk about is modified duration!

Maturity	Yield	Yesterday	Last Week	Last Month
3 Month	1.64	1.67	1.64	1.63
6 Month	1.86	1.87	1.88	1.87
2 Year	2.30	2.30	2.36	2.49
3 Year	2.15	2.14	2.17	2.35
5 Year	2.97	2.91	3.09	3.19
10 Year	3.67	3.70	3.81	3.93
30 Year	4.26	4.30	4.42	4.54



Behavior of Price Volatility

- Price volatility increases as the coupon rate decreases.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
  - Zero-coupon bonds are the most volatile.
- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility

- For bonds selling above par or at par, price volatility increases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.
  (But the *yields* of long-term bonds are less volatile than those of short-term bonds.)
- For bonds selling below par, price volatility first increases then decreases.
  - Longer maturity here cannot be equated with higher price volatility.

Figure 4.1 **(Premium bonds and par bonds):** Volatility with respect to terms to maturity

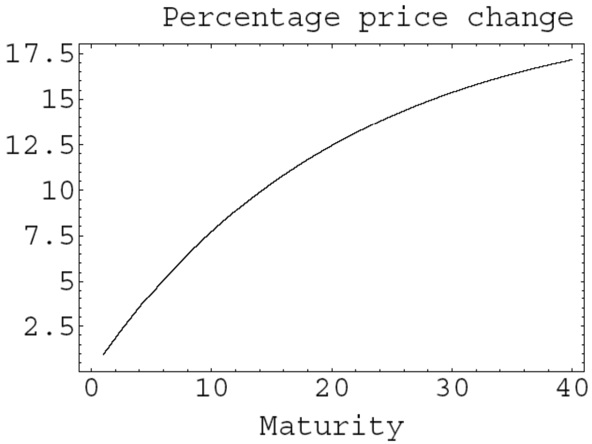
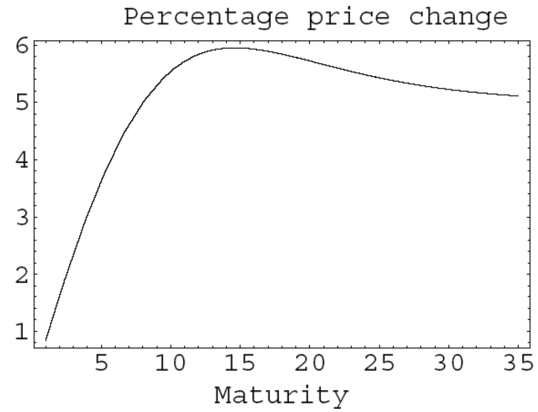


Figure 4.1 **(discount bonds):** Volatility with respect to terms to maturity.



Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price. Formally,

$$MD \equiv \frac{1}{p}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

- The Macaulay duration, in periods, is equal to

$$MD = -(1+y)\frac{\partial P/P}{\partial y} \qquad (4.2)$$

The Proof

$$\because P = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C+F}{(1+y)^n}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{-C}{(1+y)^2} + \frac{-2C}{(1+y)^3} + \ldots + \frac{-n(C+F)}{(1+y)^{n+1}}$$

$$\therefore \frac{\partial P}{\partial y} = -\frac{1}{1+y}\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}\right]$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}\right]\frac{1}{P}$$

$$Define: MD = \frac{\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}}{P} = \frac{\sum_{i=1}^{n} iC_i}{P}$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}MD \Rightarrow \frac{\partial P/P}{\partial y/1+y} = -MD$$

Example:

Duration of 6-Year Eurobond, 1,000 Face Value, 8 Percent Coupon and Market Yields 8%

t	C_t	DF_t	$C_t \times DF_t$	$C_t \times DF_t \times t$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1080	0.6302	680.58	4083.48
			1000	4992.71
MD=4992.71/1000=4.993 years				

C is cash flow, DF is discount factor

C++: Macaulay Duration的計算

- Macaulay Duration的計算

$$MD = \frac{1}{P}\left(\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}\right)$$

- 利用for loop同時求算 $\frac{1}{P}$ 和 $\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}$
- 相乘即為答案

完整程式碼

```
#include <stdio.h>
void main()
{
     int n;
     float c, r, Value=0,Discount,Duration=0;
     printf("請輸入期數:");
     scanf("%d",&n);
     printf("請輸入債息:");
     scanf("%f",&c);
     printf("請輸入利率:");
     scanf("%f",&r);
     for(int i=1;i<=n;i=i+1)
     {
      Discount=1;
      for(int j=1;j<=i;j++)
      {
       Discount=Discount/(1+r);
      }
      Duration=Duration+i*Discount*c;
      Value=Value+Discount*c;
      if(i==n)
      {
       Value=Value+Discount*100;
       Duration=Duration+n*Discount*100;
      }
     }
     Duration=Duration/Value;
     printf("Duration=%f",Duration);
}
```

For迴圈: 計算 Duration,Value

For迴圈: 計算Discount factor

If 條件式: i等於n時,考慮face value

Homework 3

- Program Exercise

課本(C++財務程式設計)第三章習題8, 9。

8.請嘗試使用一些簡單的財務知識，來驗證本章的計算存續期間的範例程式產生的答案是否合理。假定債券支付的債息為0，則其存續期間應為多少？請問當債息提高（或下降），存續期間應提高還是下降？並將推論的結果輸入範例程式中，驗證推論的結果是否和程式的輸出相符合。

Homework 3

9.在本章實例演練中討論的存續期間（duration）稱為Macaulay duration (MD)，其定義經化簡可得 $\frac{-\frac{\partial P}{\partial r}\times(1+r)}{P}$ 。為了討論債券價格的變化和殖利率變動的關係，可定義Modified duration, Modified duration 定義為 $\frac{-\partial P/P}{\partial r}=\frac{MD}{(1+r)}$ 。請修改本章計算存續期間的範例程式，計算Modified duration。請利用程式中已計算出的Modified duration，計算當殖利率變動一個basis point時，該債券價格變動的百分比。

Macaulay Duration

- The MD of a coupon bond is less than its maturity.
- The MD of a zero-coupon bond
- The MD of a Consol

MD of a Coupon Bond

- The MD of a coupon bond is

$$MD=\frac{1}{P}\left(\sum_{i=1}^{n}\frac{iC}{(1+y)^{i}}+\frac{nF}{(1+y)^{n}}\right) \qquad (4.3)$$

Where C is the period fixed interest flow.

The MD of a zero-coupon bond

- MD of a zero-coupon bond is it’s final maturity (n).
- Proof: because no cash flows before maturity, the MD is

$$MD=\frac{\sum_{i=1}^{n}iC_i(1+y)^{-i}}{\sum_{t=1}^{n}C_t(1+y)^{-i}}=\frac{nC_n(1+y)^{-n}}{C_n(1+y)^{-n}}=n$$

The MD of a Consol Bond

- A consol bond pay a fixed coupon each period but it never matures. $(Maturity\ \ date = \infty)$
- The duration of a consol bond is: $MD_c = 1 + \frac{1}{y}$

$$\because P = \frac{C}{y} \Rightarrow C = Py$$

$$\because MD = \frac{\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \frac{3C}{(1+y)^3} + \ldots.\infty}{P} = \frac{\frac{Py}{(1+y)} + \frac{2Py}{(1+y)^2} + \frac{3Py}{(1+y)^3} + \ldots.\infty}{P}$$

$$= \frac{y}{(1+y)} + \frac{2y}{(1+y)^2} + \frac{3y}{(1+y)^3} + \ldots\ldots\infty$$

$$\because \frac{1}{(1+y)} MD = \frac{y}{(1+y)^2} + \frac{2y}{(1+y)^3} + \frac{3y}{(1+y)^4} + \ldots\ldots\infty$$

$$\Rightarrow MD - \frac{1}{(1+y)} MD = \frac{y}{(1+y)} + \frac{y}{(1+y)^2} + \frac{y}{(1+y)^3} + \ldots.\infty \Rightarrow \frac{y}{(1+y)} MD = \frac{y/(1+y)}{1 - \frac{1}{(1+y)}} = 1 \Rightarrow MD = \frac{1+y}{y} = 1 + \frac{1}{y}$$

Where y is yield to maturity

The MD of Floating-rate instruments

- A floating-rate instrument makes interest rate payments based on some publicized index such as the London Interbank Offered Rate (LIBOR), the U.S. T-bill rate.
- Instead of being locked into a number, the coupon rate is reset periodically to reflect the prevailing interest rate.
- Floating-rate instrument are typically less sensitive to interest rate changes than are fixed-rate instrument.

The MD of Floating-rate instruments

- Assume that the fixed coupon rate ***c*** in first *j* period, **y** in *n-j* period, also market yield is **y** now. The first reset date is ***j*** period from now, and reset will be performed thereafter.
- Let the principal be \$1 for simplicity. The cash flow of the floating-rate instrument is

$$\underbrace{c, c, \ldots\ldots c,}_{j}\ \underbrace{y, \ldots\ldots\ldots y,}_{n-j}\ y+1$$

- The MD of a floating-rate instrument is $MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}}$

Denote the MD of an otherwise identical fixed-rate bond.

Homework 4

- Prove that

$$MD_{floating} = MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}}$$

Where the bond is priced at par ,and the principal be \$1 for simplicity.

Conversion

- To convert the MD to be year based, modify(4.3) as follow:

$$\frac{1}{p}\left(\sum_{i=1}^{n}\frac{i}{k}\frac{C}{(1+\frac{y}{k})^{i}}+\frac{n}{k}\frac{F}{(1+\frac{y}{k})^{n}}\right)$$

Where ***y*** is the *annual yield* and ***k*** is the compounding frequency per annum.

- Equation (4.2) also becomes $MD=-(1+\frac{y}{k})\frac{\partial P/p}{\partial y}$
- Note from the definition that $MD(in\ years)=\frac{MD(in\ periods)}{k}$

Difference of formulas

- Macaulay Duration: $MD=-\frac{\partial P/P}{\partial y/(1+y)}$
- Modified Duration: $D=\frac{MD}{1+y}=-\frac{\partial P/P}{\partial y}$
- Dollar Duration: $DD=D\times P=-\frac{\partial P}{\partial y}$

Effective Duration

- A general numerical formula for volatility is the effective duration, defined as

$$\frac{P_{-}-P_{+}}{P_{0}(y_{+}-y_{-})} \qquad (4.5)$$

where P_- is the price if the yield is decreased by Δy, P^+ is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, and Δy is small.

- We can compute the effective duration of just about any financial instrument.

Effective Duration

- Most useful where yield changes alter the cash flow or securities whose cash flow is so complex that simple formulas are unavailable
- Duration of a security can be longer than its maturity or negative.
  - Consider a cash flow: -1 @ time 1 and 2 @time 2
  - Consider a cap
- Neither makes sense under the maturity interpretation.
- An alternative is to use $\frac{P_0-P_+}{P_0\Delta y}$.

Immunization and MD

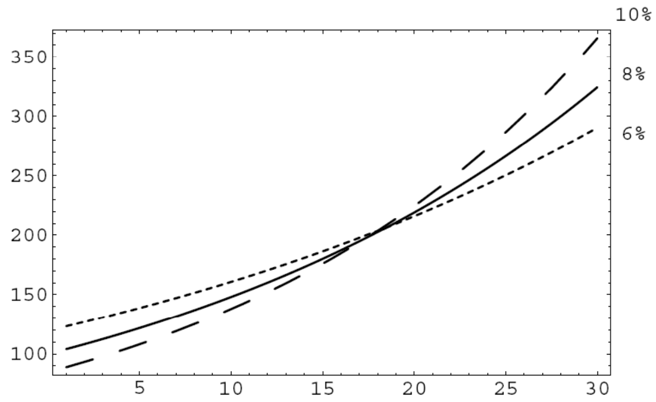
- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- How do we find such a bond portfolio?

→A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability.

– At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall.

– Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise

Bond value under three rate scenarios



Immunization

- Assume the liability is ***L*** at time ***m*** and the current interest rate is ***y***. We are looking for a portfolio such that

(1) FV is L at the horizon m;

(2) $\partial FV/\partial y = 0$;

(3) FV is convex around y.

- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean ***L*** is the portfolio's minimum **FV** at horizon for small rate changes.

The Proof (1)

- Let $FV \equiv P(1+y)^m$, where P is the PV of the portfolio. Now,

$$\frac{\partial FV}{\partial y} = m(1+y)^{m-1}P + (1+y)^m \frac{\partial P}{\partial y} \qquad (4.8)$$

- Imposing Condition (2) leads to

$$m = -(1+y)\frac{\partial P / P}{\partial y} \qquad (4.9)$$

- The MD is equal to the horizon m.

The Proof (2)

- Employ coupon bond for immunization, because

$$FV = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}}$$

- It follows that

$$\frac{\partial^2 FV}{\partial^2 y} > 0, \ for \ y > -1 \qquad (4.10)$$

- Because the FV is convex for y>1, the minimum value of FV is indeed **L**.

Example: Immunization by using duration technique

- Suppose that we are in 2007, and the insurer has to make a guarantee payment **$1,469** to a policyholder in 5 years, 2012. The amount is equivalent to investing $1,000 at an annually compound rate of 8% over 5 years.
- Strategy1: Buy five-year maturity discount bonds.
- Strategy2: Buy five-year duration coupon bonds.

Strategy1: Buy five-year maturity discount bonds

- If the insurer buy 1.469 units of these bonds at a total cost of $1000 in 2007, these investment would produce exactly $1469 on maturity in five years.
- The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability.

$$P = \frac{1000}{1.08^5} = 680.58 \Rightarrow total \ cost = 1.469 \times 680.58 = 1000$$

$$cash \ flow \ in \ five \ years = \$1000 \times 1.469 = \$1469$$

Strategy2: Buy five-year duration coupon bonds.

- The gain or losses on reinvestment income that result from an interest rate change are exactly offset by losses or gains from the bond proceeds on sale.

	YTM fall to 7%	YTM is 8%	YTM rise to 9%
Coupons (5x$80)	400	400	400
Reinvestment income	60	69	78
Sale of bond at end of the 5th year	1009	1000	991
	$1469	$1469	$1469

Cash matching

Immunization

- If there is no single bond whose MD match the horizon, a portfolio of two bonds A and B, can be assembled by the solution of

$$1 = \omega_A + \omega_B$$
$$D = \omega_A D_A + \omega_B D_B \quad (\textit{See} \text{ next page})$$

Here, $\mathbf{D_i}$ is the MD of bond i and $\boldsymbol{\omega_i}$ is the weight of bond i in the portfolio.

- Make sure that D falls between D_A and D_B to guarantee $\omega_A>0, \omega_B>0$, and positive portfolio convexity.

$$Set \;\; D = \frac{1}{P}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

$$D_A = \frac{1}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i}, \quad D_B = \frac{1}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

(A_i, B_i : cashflow of A and B at i-th period)

$$\therefore W_A D_A + W_B D_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$Set \;\; P = W_A P + W_B P$$

$$we \; can \; buy \; \frac{W_A P}{P_A} \; units \; of \; A, \; and \; \frac{W_B P}{P_B} \; units \; of \; B.$$

$$then \; D = \frac{1}{P}\sum_{i=1}^{n_A}\frac{i\frac{W_A P}{P_A}A_i}{(1+y)^i} + \frac{1}{P}\sum_{i=1}^{n_B}\frac{i\frac{W_B P}{P_B}B_i}{(1+y)^i} = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$\therefore \; D = W_A D_A + W_B D_B$$

In Class Exercise

- The liability has an MD of 3 years, but the money manager has access to only two kinds of bonds with MDs of 1 year and 4 years. What is the right proportion of each bond in the portfolio in order to match the liability's MD?

Limitations of Duration

- Duration matching can be costly.
- Immunization is a dynamic problem.
  - Because continuous rebalancing may not be easy to do and involves costly transaction fees.
  - There is a trade-off between being perfectly immunized and the transaction costs of maintaining.
- Large interest rate and convexity (see next figure).
  - Duration accurately measures the price sensitivity of fixed-income securities for small change in interest rates.

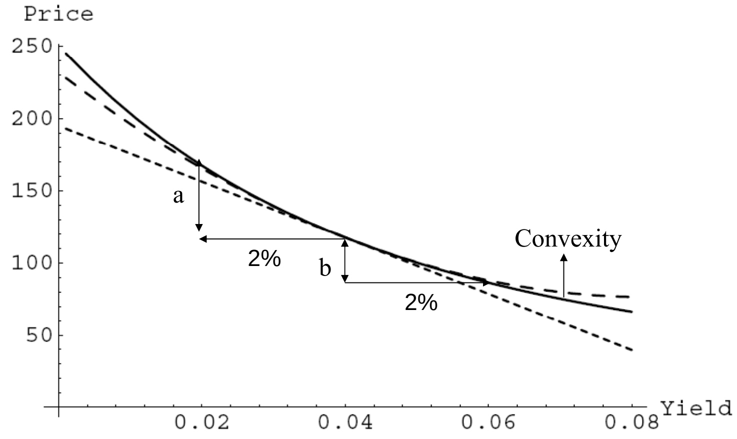
Convexity

- Convexity is defined as

$$convexity\,(in\ period) \equiv \frac{\partial^2 P}{\partial y^2}\frac{1}{P} \qquad (4.14)$$

- The convexity of a coupon bond is positive.
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Between two bonds with the same duration, the one with a higher convexity is more valuable.

Figure 4.6: Linear and quadratic approximation to bond price changes.



Convexity

- The approximation $\Delta P/P \approx -$ (modified) duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 = -duration \times \Delta y + \frac{1}{2} \times convexity \times (\Delta y)^2$$

Calculation of Convexity

- Formula:

$$CX = \text{Scaling factor}(\text{The capital loss from 1bp rise} + \text{The capital gain from 1bp fall})$$
$$= 10^8\left(\frac{\Delta P^-}{P} + \frac{\Delta P^+}{P}\right)$$

- Example: To calculate convexity of the 8 percent coupon, 8 percent yield, six-year maturity Eurobond that had a price of $1000:

$$CX = 10^8\left(\frac{999.53785-1000}{1000} + \frac{1000.46243-1000}{1000}\right)$$
$$= 10^8(0.00000028) = 28$$

Example

- Given convexity C, the percentage price change expressed in percentage terms is approximated by $-D \times \Delta r + C \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
- For example, if $D = 10$, $C = 150$, and $\Delta r = 2\%$, price will drop by 17% because

$\Delta P/P = -10 \times 2\% + 1/2 \times 150 \times (2\%)^2 = -17\%$

In Class Exercise

- Show that the convexity of a n-period zero-coupon bond is $n(n+1)/(1+y)^2$

Immunization (barbell Portfolio)

- Two bond portfolios with varying duration pairs D_A, D_B can be assembled to satisfy $D = \omega_A D_A + \omega_B D_B$
  However, which one is to be preferred?
- Let there be n kinds of bonds, with bond i having duration D_i and convexity C_i, where $D_1 < D_2 < \ldots < D_n$. We then solve the follow constrained optimization problem:

$$\begin{aligned} \text{maximize} \quad & \omega_1 C_1 + \omega_2 C_2 + \ldots\ldots + \omega_n C_n \\ \text{subject to} \quad & \omega_1 + \omega_2 + \ldots\ldots \omega_n = 1 \\ & \omega_1 D_1 + \omega_2 D_2 + \ldots\ldots + \omega_n D_n = D \end{aligned}$$

The solution usually implies a barbell portfolio, which consists of very short-term bonds and very long-term bonds..

Proof of Convexity

$$C = W_A C_A + W_B C_B$$

$$\frac{\partial^2 P}{\partial y^2} = \sum_{i=1}^{n} \frac{i(i+1)C_i}{(1+y)^{i+2}} \qquad Convexity = \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = \frac{1}{P}\sum_{i=1}^{n} \frac{i(i+1)C_i}{(1+y)^{i+2}} \equiv C$$

(Let A_i, B_i : cash flow of A and B at i-th period)

$$C_A = \frac{1}{P_A}\sum_{i=1}^{n_A} \frac{i(i+1)A_i}{(1+y)^{i+2}}, \quad C_B = \frac{1}{P_B}\sum_{i=1}^{n_B} \frac{i(i+1)B_i}{(1+y)^{i+2}}$$

$$\therefore W_A C_A + W_B C_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A} \frac{i(i+1)A_i}{(1+y)^{i+2}} + \frac{W_B}{P_B}\sum_{i=1}^{n_B} \frac{i(i+1)B_i}{(1+y)^{i+2}}$$

$$Set\ P = W_A P + W_B P \quad we\ can\ buy\ \frac{W_A P}{P_A} units\ of\ A,\ and\ \frac{W_B P}{P_B} units\ of\ B$$

$$then\ C = \frac{1}{P}\sum_{i=1}^{n_A} \frac{i(i+1)\frac{W_A P}{P_A}A_i}{(1+y)^{i+2}} + \frac{1}{p}\sum_{i=1}^{n_B} \frac{i(i+1)\frac{W_B P}{P_B}B_i}{(1+y)^{i+2}}$$

$$= \frac{W_A}{P_A}\sum_{i=1}^{n_A} \frac{i(i+1)A_i}{(1+y)^{i+2}} + \frac{W_B}{P_B}\sum_{i=1}^{n_B} \frac{i(i+1)B_i}{(1+y)^{i+2}} = W_A C_A + W_B C_B$$

Lagrange Multiplier Method

$$
\begin{aligned}
&\textit{function } f(x_1, x_2, ..., x_n) \\
&\textit{subject to } g(x_1, x_2, ..., x_n) = 0 \\
&F(x_1, x_2, ..., x_n, \lambda) = f(x_1, x_2, ..., x_n) + \lambda \bullet g(x_1, x_2, ..., x_n) \\
&Fx_1(x_1, x_2, ..., x_n, \lambda) = 0 \\
&Fx_2(x_1, x_2, ..., x_n, \lambda) = 0 \\
&\vdots \\
&Fx_n(x_1, x_2, ..., x_n, \lambda) = 0 \\
&g(x_1, x_2, ..., x_n) = 0
\end{aligned}
$$

A Simple Example

$$
\begin{aligned}
&\min \quad f(x, y) = 5x^2 + 6y^2 - xy \\
&s.t \qquad x + 2y = 24
\end{aligned}
$$

$$g(x, y) = x + 2y - 24 = 0$$

$$
\begin{aligned}
&F(x, y, \lambda) = 5x^2 + 6y^2 - xy + \lambda(x + 2y - 24) \\
&F_x(x, y, \lambda) = 10x - y + \lambda = 0....(1) \\
&F_y(x, y, \lambda) = 12y - x + 2\lambda = 0..(2) \\
&g = x + 2y - 24 = 0..................(3)
\end{aligned}
$$

根據(1)、(2) 得$x = \frac{2}{3}y$

$x = \frac{2}{3}y$ 代入(3),得$y = 9, x = 6$

Use Lagrange Multiplier Method to obtain the optimal bond portfolio

$$
\begin{aligned}
&\max \ \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n \\
&s.t \quad \omega_1 + \omega_2 + \ldots + \omega_n = 1 \\
&\qquad \omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n = D \\
&g_1(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 + \omega_2 + \ldots + \omega_n - 1 = 0 \\
&g_2(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D = 0 \\
&F(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n + \\
&\qquad \lambda_1(\omega_1 + \omega_2 + \ldots + \omega_n - 1) + \lambda_2(\omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D)
\end{aligned}
$$

Example: Immunization (Convexity is desirable)

- Consider a pension fund manger with a 15-year payout horizon. To immunize the risk of interest rate changes, the manger purchase bonds with a 15-year duration. Consider two alternative strategies to achieve this:
- Strategy1: Invest 100 percent of resources in a 15-year deep-discount bond with an 8 percent yield. (Bullet portfolio)
- Strategy2: Invest 50 percent in the very short-term money market and 50 percent in 30-year deep-discount bond with an 8 percent yield. (Barbell portfolio)

Example: Immunization (Convexity is desirable)

- Strategy1:
  Duration =15
  Convexity =206
  value of the convexity =1/2 × convexity × $\triangle y^2$= 25.75%

△y=5%

- Strategy2:
  Duration =1/2 × 0 + 1/2 × 30 =15
  Convexity =1/2 × 0+1/2 × 797 =398.5
  Value of the convexity =1/2 × convexity × $\triangle y^2$ = 49.81%

High convexity is more valuable

- The manger may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as a result, both positive and negative shocks to interest rates would have beneficial effects on the net worth.

Categories of Immunization

- Cash matching
- Rebalancing

Cash matching

- Cash matching is the approach that a stream of liability can always be immunized with a matching stream of zero-coupon bonds.
- Two problem with this approach are that (1) zero-coupon bonds may be missing for certain matruity.(2) they typically carry lower yield.
- Recall example (Immunization by using duration technique).

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.
- The MD decreases as time passes.
- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  – Consider a coupon bond whose MD matches horizon.
  – Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.
  – So immunization needs to be reestablished even if interest rates never change.

Hedging

- Hedging aims to offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$DD \equiv \text{modified duration} \times \text{price}(\%\text{ of par}) = -\frac{\partial P}{\partial y}$$

- The approximate dollar price change per \$100 of par value is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}$$

Hedging

- Because securities may react to interest rate changes differently, we define yield beta to measure relative yield changes.

$$\text{yield beta} \equiv \frac{\text{change in yield for the hedged security}}{\text{change in yield for the hedging security}}$$

- Let the hedge ratio be

$$h \equiv \frac{\text{dollar duration of the hedged security}}{\text{dollar duration of the hedging security}} \times \text{yield beta} \qquad (4.13)$$

- Then hedging is accomplished when the value of the hedging security is h times that of the hedged security.

Example 4.2.2

- Suppose we want to hedge bond A with a duration of seven by using bond B with a duration of eight. Under the assumption that yield beta is one and both bonds are selling at par, the hedge ratio is 7/8, This means that an investor who is long \$1 million of bond A should short \$7/8 million of bond B.