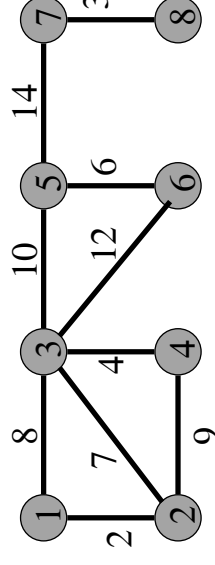


Minimum-Cost Spanning Tree

- weighted connected undirected graph
- spanning tree
- cost of spanning tree is sum of edge costs
- find spanning tree that has minimum cost

Example



- Network has 10 edges.
- Spanning tree has only $n - 1 = 7$ edges.
- Need to either select 7 edges or discard 3.

Edge Selection Strategies

- Start with an n -vertex 0-edge forest. Consider edges in ascending order of cost. Select edge if it does not form a cycle together with already selected edges.
 - Kruskal's method.
- Start with a 1-vertex tree and grow it into an n -vertex tree by repeatedly adding a vertex and an edge. When there is a choice, add a least cost edge.
 - Prim's method.

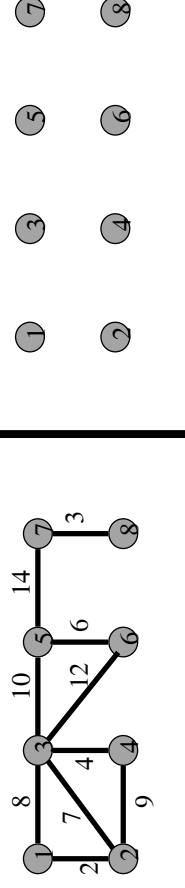
Edge Selection Strategies

- Start with an n -vertex forest. Each component/tree selects a least cost edge to connect to another component/tree. Eliminate duplicate selections and possible cycles. Repeat until only 1 component/tree is left.
 - Sollin's method.

Edge Rejection Strategies

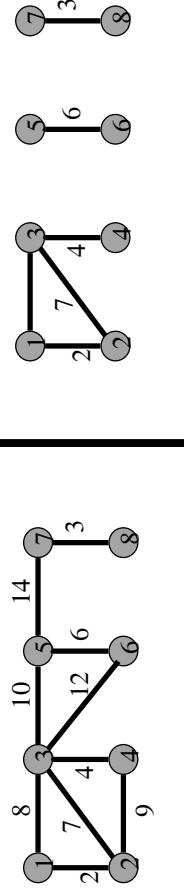
- Start with the connected graph. Repeatedly find a cycle and eliminate the highest cost edge on this cycle. Stop when no cycles remain.
- Consider edges in descending order of cost. Eliminate an edge provided this leaves behind a connected graph.

Kruskal's Method



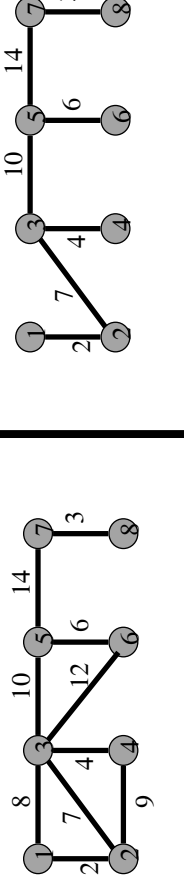
- Start with a forest that has no edges.
- Consider edges in ascending order of cost.
- Edge (1,2) is considered first and added to the forest.

Kruskal's Method



- Edge (7,8) is considered next and added.
- Edge (3,4) is considered next and added.
- Edge (5,6) is considered next and added.
- Edge (2,3) is considered next and rejected because it creates a cycle.
- Edge (1,3) is considered next and rejected because it creates a cycle.

Kruskal's Method



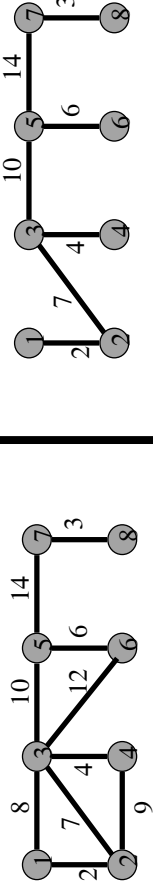
- Edge (2,4) is considered next and rejected because it creates a cycle.
- Edge (3,5) is considered next and added.
- Edge (3,6) is considered next and rejected.
- Edge (5,7) is considered next and added.

Kruskal's Method



- $n - 1$ edges have been selected and no cycle formed.
- So we must have a spanning tree.
- Cost is 46.
- Min-cost spanning tree is unique when all edge costs are different.

Prim's Method



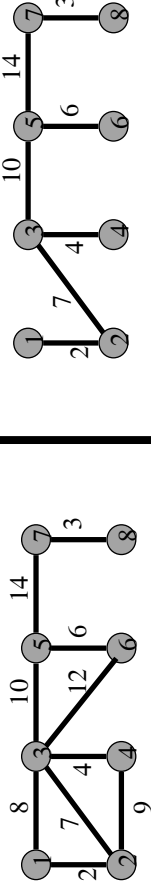
- Start with any single vertex tree.
- Get a 2-vertex tree by adding a cheapest edge.
- Get a 3-vertex tree by adding a cheapest edge.
- Grow the tree one edge at a time until the tree has $n - 1$ edges (and hence has all n vertices).

Sollin's Method



- Start with a forest that has no edges.
- Each component selects a least cost edge with which to connect to another component.
- Duplicate selections are eliminated.
- Cycles are possible when the graph has some edges that have the same cost.

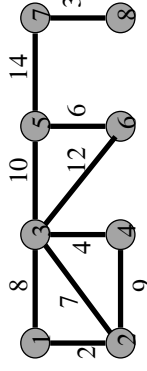
Sollin's Method



- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.

In Class Exercise

- Generate a min. spanning tree by the rule:
 - Start with the connected graph. Repeatedly find a cycle and eliminate the highest cost edge on this cycle. Stop when no cycles remain.



Pseudocode For Kruskal's Method

```
Start with an empty set T of edges.
while (E is not empty && |T| != n-1)
{
  Let (u,v) be a least-cost edge in E.
  E = E - {(u,v)}. // delete edge from E
  if ((u,v) does not create a cycle in T)
    Add edge (u,v) to T.
}
if (|T| == n-1) T is a min-cost spanning tree.
else Network has no spanning tree.
```

Minimum-Cost Spanning Tree Methods

- Can prove that all stated edge selection/rejection result in a minimum-cost spanning tree.
- Prim's method is fastest.
 - $O(n^2)$ using an implementation similar to that of Dijkstra's shortest-path algorithm. (See Lecture 24)
 - $O(e + n \log n)$ using a Fibonacci heap. (See Chap 9)
- Kruskal's uses union-find trees to run in $O(n + e \log e)$ time.

Data Structures For Kruskal's Method

Edge set E.

Operations are:

- Is E empty?
- Select and remove a least-cost edge.

Use a min heap of edges.

- Initialize. $O(e)$ time.
- Remove and return least-cost edge. $O(\log e)$ time.

Data Structures For Kruskal's Method

Set of selected edges T .

Operations are:

- Does T have $n - 1$ edges?
- Does the addition of an edge (u, v) to T result in a cycle?
- Add an edge to T .

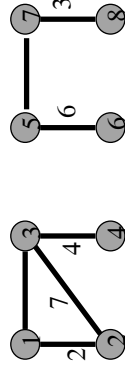
Data Structures For Kruskal's Method

Use an array for the edges of T .

- Does T have $n - 1$ edges?
 - Check number of edges in array. $O(1)$ time.
- Does the addition of an edge (u, v) to T result in a cycle?
 - Discussed later.
- Add an edge to T .
 - Add at right end of edges in array. $O(1)$ time.

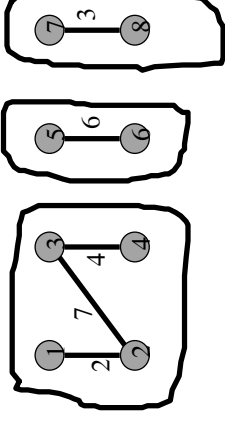
Data Structures For Kruskal's Method

Does the addition of an edge (u, v) to T result in a cycle?



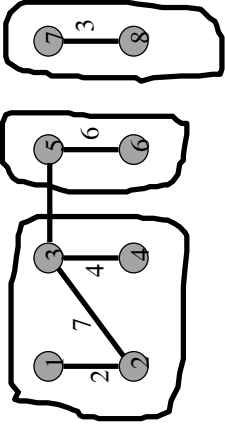
- Each component of T is a tree.
- When u and v are in the same component, the addition of the edge (u,v) creates a cycle.
- When u and v are in the different components, the addition of the edge (u,v) does not create a cycle.

Data Structures For Kruskal's Method



- Each component of T is defined by the vertices in the component.
- Represent each component as a set of vertices.
 - $\{1, 2, 3, 4\}$, $\{5, 6\}$, $\{7, 8\}$
- Two vertices are in the same component iff they are in the same set of vertices.

Data Structures For Kruskal's Method



- When an edge (u, v) is added to T , the two components that have vertices u and v combine to become a single component.
- In our set representation of components, the set that has vertex u and the set that has vertex v are united.
 - $\{1, 2, 3, 4\} + \{5, 6\} \Rightarrow \{1, 2, 3, 4, 5, 6\}$

Data Structures For Kruskal's Method

- Initially, T is empty.
 - ① ③ ⑤ ⑦
 - ② ④ ⑥ ⑧
- Initial sets are:
 - $\{1\} \{2\} \{3\} \{4\} \{5\} \{6\} \{7\} \{8\}$
- Does the addition of an edge (u, v) to T result in a cycle? If not, add edge to T .
 - $s1 = \text{Find}(u); s2 = \text{Find}(v);$
 - if $(s1 \neq s2)$ Union($s1, s2$);

Data Structures For Kruskal's Method

- Use fast solution for disjoint sets.
- Initialize.
 - $O(n)$ time.
- At most $2e$ finds and $n-1$ unions.
 - Very close to $O(n + e)$.
- Min heap operations to get edges in increasing order of cost take $O(e \log e)$.
- Overall complexity of Kruskal's method is $O(n + e \log e)$.

Homework

- Exercise 1 @P359