Exercise 1.14 (Change of measure for an exponential random variable). Let X be a nonnegative random variable defined on a probability space (Ω, F, P) with the *exponential distribution*, which is

$$\mathrm{P}\{X \le a\} = 1 - e^{-\lambda a}, a \ge 0$$

where λ is a positive constant. Let $\tilde{\lambda}$ be another positive constant, and define

$$Z = \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X}$$

Define $\tilde{\mathrm{P}}$ by

$$\tilde{\mathrm{P}}(A) = \int_A Z \, d\mathrm{P} \qquad \text{for all } A \in F$$

(i) Show that $\tilde{\mathrm{P}}(\Omega) = 1$.

(ii) Compute the cumulative distribution function for the random variable X under the probability measure $\tilde{\mathrm{P}}$.

※(Thm 1.6.7 Randon-Nikody'm)

$\tilde{\mathrm{P}}$ and P are equivalent probability measures defined on (Ω, F).

X:exponencial disrtibution (λ)

$f(x) = \lambda e^{-\lambda x}, \mathrm{E}[X] = \dfrac{1}{\lambda}$

Z : almost surely positive (∵ exponential function)

$$E[Z] = \int_0^\infty \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} dP(x) = \int_0^\infty \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \lambda e^{-\lambda x} dx = \int_0^\infty \tilde{\lambda} e^{-\tilde{\lambda} x} dx \sim \exp(\tilde{\lambda})$$

$\therefore E[Z] = 1$

(i)

$$\tilde{\mathrm{P}}(\Omega) = \int_\Omega Z \, d\mathrm{P} = E[Z] = 1$$

(ii)

$$\tilde{\mathrm{P}}\{X \le a\} = \int_{x \le a} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} dP(x) = \int_0^a \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \lambda e^{-\lambda x} dx$$

$$= \int_0^a \tilde{\lambda} e^{-\tilde{\lambda} x} dx = -e^{-\tilde{\lambda} x} \Big|_0^a = 1 - e^{-\tilde{\lambda} a} \qquad \forall a \ge 0$$