Stochastic Calculus for Finance II Continuous-Time Models
Chapter 2 Exercise

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Exercise 2.5 Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{2|x|+y}{\sqrt{2\pi}}\exp\{-\dfrac{(2|x|+y)^2}{2}\} & \text{if } y\ge -|x| \\ 0 & \text{if } y<-|x|\end{cases}$$

Show that X and Y are standard normal variables and that they are uncorrelated but not independent.

Ans.

Show X and Y are standard normal random variables:

$$f_X(x,y)=\int_{-|x|}^{\infty} f_{X,Y}(x,y)dy=\frac{1}{\sqrt{2\pi}}e^{-\frac{4|x|^2}{2}}\{\int_{-|x|}^{\infty}(2|x|+y)e^{-(2|x|y+\frac{y^2}{2})}dy\}$$
$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{4|x|^2}{2}}\{\ -e^{-(2|x|y+\frac{y^2}{2})}\Big|_{-|x|}^{\infty}\ \}=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ x\in(-\infty,\infty)$$

$\therefore X \sim N(0,1)$

$$f_Y(x,y)=\int_{-y}^{\infty} f_{X,Y}(x,y)dx+\int_{-\infty}^{y} f_{X,Y}(x,y)dx=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}},\ y\in(-\infty,\infty)$$

(x>0) (x ≤ 0)

$\therefore Y \sim N(0,1)$

Show they are uncorrelated but not independent:

uncorrelated:

$Cov(X,Y)=E[(X-0)(Y-0)]=E[XY]$

$$E[XY]=\int_{-\infty}^{\infty}\int_{-|x|}^{\infty} xyf_{X,Y}(x,y)dydx=\int_{0}^{\infty}\int_{-x}^{\infty} xyf_{X,Y}(x,y)dydx+\int_{-\infty}^{0}\int_{x}^{\infty} xyf_{X,Y}(x,y)dydx$$

part A (x>0) part B (x ≤ 0)

part A= $\int_{0}^{\infty}\int_{-x}^{\infty} xy\frac{2x+y}{\sqrt{2\pi}}e^{-\frac{(2x+y)^2}{2}}dydx$

part B= $\int_{-\infty}^{0}\int_{x}^{\infty} xy\frac{-2x+y}{\sqrt{2\pi}}e^{-\frac{(-2x+y)^2}{2}}dydx$

let $z=-x, dz=-dx$

part A= $-\int_{-\infty}^{0}\int_{z}^{\infty} zy\frac{-2z+y}{\sqrt{2\pi}}e^{-\frac{(-2z+y)^2}{2}}dydz=-$part B

$\therefore E[XY]=$ part A+part B=0 $\Rightarrow Cov(X,Y)=0$

not independent:

$\because f_X(x,y)f_Y(x,y)=\frac{1}{2\pi}e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}} \neq f_{X,Y}(x,y)$

$\therefore$ X and Y are not independent.

※ Construct a Markovian process but not a martingale.

$$\mathrm{X}_i \overset{iid}{\sim} X = \begin{cases} 1, prob. = 0.5 \\ 0, prob. = 0.5 \end{cases}$$

let $S_n = \sum_{i=1}^{n} \mathrm{X}_i$

$$\mathrm{E}[S_N \mid F_n] = \mathrm{E}[\sum_{i=1}^{N} \mathrm{X}_i \mid F_n] = \mathrm{E}[\sum_{i=1}^{n} \mathrm{X}_i + \sum_{i=n+1}^{N} \mathrm{X}_i \mid F_n] = \sum_{i=1}^{n} \mathrm{X}_i + \mathrm{E}[\sum_{i=n+1}^{N} \mathrm{X}_i \mid F_n]$$

$$= S_n + \sum_{i=n+1}^{N} \mathrm{E}[\mathrm{X}_i \mid F_n] > S_n \quad (\because \mathrm{E}[\mathrm{X}_i \mid F_n] = \mathrm{E}[\mathrm{X} \mid F_n] = \frac{1}{2} > 0)$$

$\therefore S_N$ is not a martingale.

$\forall f, \exists g$ st. $\mathrm{E}[f(S_N) \mid F_n] = \mathrm{E}[f(\sum_{i=1}^{n} \mathrm{X}_i + \sum_{i=n+1}^{N} \mathrm{X}_i) \mid F_n]$ let $Y = \sum_{i=n+1}^{N} \mathrm{X}_i$ is independent of F_n

$$= \sum_{y} f(y + \sum_{i=1}^{n} \mathrm{X}_i) h_Y(y) = g(\sum_{i=1}^{n} \mathrm{X}_i) = g(S_n)$$

$\therefore S_N$ is a Markovian process.