Option Pricing, Hedging, and Efficient Monte Carlo Methods

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Black-Scholes Model

Under the physical probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, there are two assets within an economy:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \text{ (stock)} \\ dr_t &= r dt. \text{ (bond)} \end{aligned}$$

- μ: rate of returns.
- r: risk-free interest rate.
- σ: volatility (constant).
- W_t: 1-d. standard Brownian Motion.

Black-Scholes Theory

The European option price is

$$P(t, S_t) = \mathbb{E}^\star \left\{ e^{-r(T-t)} H(S_T) | \mathcal{F}_t \right\}$$

defined on the risk-neutral pricing probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^\star)$ such that

$$dS_t = r S_t dt + \sigma S_t dW_t^\star.$$

Perfect replication of the discounted payoff:

$$\begin{aligned} P(0, S_0) &= e^{-rT} H(S_T) \\ &\quad - \int_0^T e^{-rs} \underbrace{\frac{\partial P}{\partial x}(s, S_s)}_{Delta} \sigma S_s dW_s^\star. \end{aligned}$$

Black-Scholes Formula

Typical payoff functions are nonlinear like
$H(x) = \max\{x - K, 0\} = (x - K)^+$ a **call**.
$H(x) = \max\{K - x, 0\} = (K - x)^+$ a **put**.
K is the **strike** price.
The celebrated BS formula for the **European** call option price is

$$P(t, S_t = x) = x\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

where $d_1 = \frac{\ln(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

Some Empirical Evidence

(1) S. Alizadeh, M. Brandt, and F. Diebold, "Range-based estimation of stochastic volatility models," Journal of Finance, 57, 1047-1091, 2002.
(2) M. Chernov, R. Gallant, E. Ghysels, and G. Tauchen, "Alternative models for stock price dynamics," Journal of Econometrics, 2003, vol. 116, issue 1-2, pages 225-257.
(3) $\cdots$

Recast Financial Problems

Under generalized models,

- Pricing: No closed-form solution.

  $$P_t = \mathbb{E}^{\star}\left\{e^{-r(T-t)}H(S_T)|\mathcal{F}_t\right\}.$$

  Monte Carlo is a good computational *option.*

- Hedging: No perfect (self-financing) replication.

  Is Delta hedge enough? What Delta?

Multifactor Stochastic Volatility Model

Under a risk-neutral prob. meas. $\mathbb{P}^{\star(\Lambda)}$, a multifactor SV model is of the form:

$$
\begin{aligned}
dS_t &= rS_t dt + f(Y_t, Z_t) S_t dW_t^{(0)\star}, \\
dY_t &= c_1(Y_t, Z_t) dt + g_1(Y_t, Z_t) dW_t^{(1)\star}, \\
dZ_t &= c_2(Y_t, Z_t) dt + g_2(Y_t, Z_t) dW_t^{(2)\star}
\end{aligned}
$$

$$
\begin{aligned}
d\langle W^{(0)}, W^{(1)} \rangle_t &= \rho_1 dt \\
d\langle W^{(0)}, W^{(2)} \rangle_t &= \rho_2 dt \\
d\langle W^{(1)}, W^{(2)} \rangle_t &= \rho_{12} dt.
\end{aligned}
$$

Martingale Representation

Risky asset dynamics:

$$dS_t \;=\; rS_t dt + \sigma_t S_t dW_t^\star,$$

σ_t is a diffusion process.

$P(0, S_0, \sigma_0) = e^{-rT} H(S_T) - \mathcal{M}(P) + Martingales$

with $\mathcal{M}(P) = \int_0^T e^{-rs} \frac{\partial P}{\partial x}(s, S_s, \sigma_s) \sigma_s S_s dW_s^\star$ is a zero-centered martingale.

Additional *martingales* are related to non-tradable risks and perhaps difficult to compute.

Monte Carlo Pricing with Control Variate

$$P(0, S_0, \sigma_0) \approx \frac{1}{N} \sum_{i=1}^{N} \left[e^{-rT} H(S_T^{(i)}) - \mathcal{M}^{(i)}(\tilde{P}) \right],$$

where $\mathcal{M}(\tilde{P}) = \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial x}(s, S_s, \sigma_s) \sigma_s S_s dW_s^\star$ is a *martingale* with $\tilde{P}$ being an approximation of P.

Control by Hedging Portfolio

Clewlow and Carverhill* used hedging portfolio as a control s.t.

$$P(0, S_0, \sigma_0) \approx \frac{1}{N} \sum_{i=1}^{N} \left[e^{-rT} H(S_T^{(i)}) - \mathcal{M}^{(i)}(P_{BS}) \right],$$

with $P_{BS}(t, S_t) = P_{BS}(t, S_t; \widehat{\sigma})$. The choice of $\widehat{\sigma}$ depends on the **long-run mean** of the driving volatility process.

*Clewlow, L. and Carverhill, A. (1994) On the simulation of contingent claims, Journal of Derivatives 2:66-74.

Diffusion Operator Integral Method

Heath and Platen* proposed to use the option price approximated from deterministic volatility by removing its random source.

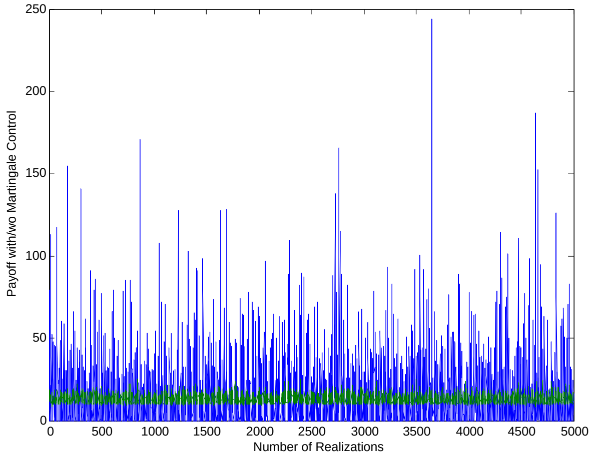
*Heath, D. and Platen, E. (2002) A variance reduction technique based on integral representations, Quantitative Finance 2:362-369.

Homogenization Method

Fouque and H.* use the homogenized option price $P_{BS}(t, S_t; \bar{\sigma})$ to construct a martingale control $\mathcal{M}^{(i)}(P_{BS})$.
The **effective** variance $\bar{\sigma}^2$ is defined as the averaging of the variance function w.r.t. an invariant distribution of a volatility process.

*Fouque, J.P. and Han (2004) A control variate method to evaluate option prices under multi-factor stochastic volatility models. Submitted.

Evaluate European Option by Martingale Control Variate



blue: Basic MC samples

green: MC+CV samples

A Variance Problem

A martingale control Variate method ends up to compute

$$P(0, S_0, Y_0, Z_0) \approx \frac{1}{N} \sum_{i=1}^{N} \left[e^{-rT} H(S_T^{(i)}) - \mathcal{M}^{(i)}(\tilde{P}) \right],$$

where the *martingale* is defined by

$$\mathcal{M}(\tilde{P}) = \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial x}(s, S_s, Y_s, Z_s) \sigma_s S_s dW_s^\star$$

with $\tilde{P}$ being an approx. to P.

Q: **Can we estimate** $Var(e^{-rT} H(S_T) - \mathcal{M}(\tilde{P}))$?

Time-Scaled Stochastic Volatility model

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_{0t}^\star \\
\sigma_t &= f(Y_t, Z_t) \\
dY_t &= \left[\frac{1}{\varepsilon} c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t)\right] dt \\
&+ \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \left(\rho_1 dW_{0t}^\star + \sqrt{1-\rho_1^2} dW_{1t}^\star\right) \\
dZ_t &= \left[\delta c_2(Z_t) + \sqrt{\delta} g_2(Z_t) \Lambda_2(Y_t, Z_t)\right] dt + \sqrt{\delta} g_2(Z_t) \\
& \cdot \left(\rho_2 dW_{0t}^\star + \rho_{12} dW_{1t}^\star + \sqrt{1-\rho_2^2-\rho_{12}^2} dW_{2t}^\star\right)
\end{aligned}
$$

A European option price is defined by

$$
P^{\varepsilon,\delta}(t,x,y,z) = \mathbb{E}^\star{}_{t,x,y,z}\left\{e^{-r(T-t)} H(S_T)\right\}.
$$

Variance Decomosition
Case: No Correlation between BMs

$$Var\left(e^{-rT}H(S_T) - \mathcal{M}(P_{BS})\right) = \mathbb{E}^{\star}{}_{0,x,y,z}\Big\{$$

$$\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s, Z_s) f^2(Y_s, Z_s) S_s^2$$

$$+\frac{1}{\varepsilon}\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial y}\right)^2 (s, S_s, Y_s, Z_s) g_1^2(Y_s) ds$$

$$+\,\delta\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)^2 (s, S_s, Y_s, Z_s) g_2^2(Z_s) ds\Big\}.$$

Variance Analysis

Under some smooth and bound conditions,

1.$\mathbb{E}^{\star}\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^2 f^2(Y_s,Z_s)S_s^2 ds\right\} < C\max\{\varepsilon,\delta\}$

2.$\mathbb{E}^{\star}\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial y}\right)^2 g_1^2(Y_s)ds\right\} \le C\,\varepsilon^2$

3.$\mathbb{E}^{\star}\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)^2 g_2^2(Z_s)ds\right\} \le C$

such that

$$Var\;\left(e^{-rT}H(S_T)-\mathcal{M}(P_{BS})\right) \le C\max\{\varepsilon,\delta\}.$$

First Inequality

By Cauchy-Schwartz inequality

$$\mathbb{E}^\star\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^2 f^2(Y_s,Z_s)S_s^2\,ds\right\}$$

$$\leq \sqrt{\mathbb{E}^\star\left\{\int_0^T \left(e^{-rs}S_s\right)^4 f^4(Y_s,Z_s)\,ds\right\}}$$

$$\times\sqrt{\mathbb{E}^\star\left\{\int_0^T \left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}-\frac{\partial P_{BS}}{\partial x}\right)^4 ds\right\}}$$

Delta Approximation

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_t}(t, S_t, Y_t, Z_t) = \mathbb{E}^{\star}{}_t \left\{ e^{-r(T-t)} \mathbf{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_t} \right\}$$

$$= \tilde{E}_t \left\{ \mathbf{I}_{\{S_T > K\}} \right\}$$

where the Radon-Nikodym derivative is

$$\frac{d\tilde{P}}{d\mathbb{P}^{\star}} = e^{-\int_0^T \frac{\sigma_t^2}{2} dt + \int_0^T \sigma_t dW_t^{(0)\star}}.$$

Digital option approximation yields

$$\left| \left(\frac{\partial P^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}}{\partial x} \right) (t, S_t, Y_t, Z_t) \right| \leq C \max\{\sqrt{\varepsilon}, \sqrt{\delta}\}.$$

Second Inequality

$$\int_0^T e^{-2rs} \left(\frac{\partial P^{\varepsilon,\delta}}{\partial y} \right)^2 (s, S_s, Y_s, Z_s) g_1^2(Y_s) ds \le C\, \varepsilon^2.$$

Conditional on the volatility process,

$$P^{\varepsilon,\delta}(t,x,y,z) = \mathbb{E}^\star{}_t \left\{ P_{BS} \left(t, x; K, T; \sqrt{\bar{\sigma}_{av}^2} \right) \right\},$$

where the realized variance is denoted by $\bar{\sigma}_{av}^2$:

$$\bar{\sigma}_{av}^2(y,z) = \frac{1}{T-t} \int_t^T f(Y_s, Z_s)^2 ds.$$

Chain Rules

$$\frac{\partial P^{\varepsilon,\delta}}{\partial y} = \mathbb{E}^{\star}{}_t \left\{ \frac{\partial P_{BS}}{\partial \sigma} \left(t, x; K, T; \sqrt{\bar{\sigma}^2_{av}(y,z)} \right) \frac{\partial \sqrt{\bar{\sigma}^2_{av}}}{\partial y} \right\}.$$

$$\frac{\partial \sqrt{\bar{\sigma}^2_{av}}}{\partial y} = \frac{\int_t^T \left[\frac{\partial f}{\partial y}(Y_s, Z_s) \frac{\partial Y_s}{\partial y} + \frac{\partial f}{\partial z}(Y_s, Z_s) \frac{\partial Z_s}{\partial y} \right] f(Y_s, Z_s) ds}{(T-t)\sqrt{\bar{\sigma}^2_{av}}}.$$

How fast does $\left(\frac{\partial Y_s}{\partial y} \quad \frac{\partial Z_s}{\partial y} \right)$ grow?

Perturbed Dynamical System

Rescaling $\tilde{Y}_s^\varepsilon = Y_{s\varepsilon}$ and $\tilde{Z}_s^\varepsilon = Z_{s\varepsilon}$, we deduce

$$\frac{d}{ds}\begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix}$$

$$+ \quad \sqrt{\varepsilon} \cdot \begin{pmatrix} \nu_1\sqrt{2}\frac{\partial \tilde{\Lambda}_1}{\partial y}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) & \frac{\partial \tilde{\Lambda}_1}{\partial z}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \\ \sqrt{\delta}\nu_2\frac{\partial \tilde{\Lambda}_2}{\partial y}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) & -\delta + \sqrt{\delta}\nu_2\sqrt{2}\frac{\partial \tilde{\Lambda}_2}{\partial z}(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{\partial \tilde{Y}_s^\varepsilon}{\partial y} \\ \frac{\partial \tilde{Z}_s^\varepsilon}{\partial y} \end{pmatrix} \text{ with } \left(\frac{\partial \tilde{Y}_0^\varepsilon}{\partial y}, \frac{\partial \tilde{Z}_0^\varepsilon}{\partial y}\right)^T = (1,0)^T.$$

Stability Theory

By a classical stability result,* we obtain $|\frac{\partial Y_s}{\partial y}| < C_1 e^{-(s-t)/\varepsilon}$ and $|\frac{\partial Z_s}{\partial y}| < C_2 \delta$ for some constants C_1 and C_2.

*R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, 1953.

Third Inequality

$$\int_0^T e^{-2rs} \left(\frac{\partial P^{\varepsilon,\delta}}{\partial z} \right)^2 (s, S_s, Y_s, Z_s) g_2^2(Z_s) ds \leq C$$

The proof is similar as in the second inequality.

Replication Error and Variance Reduction *

1. European Options

2. Barrier Options

3. American Options

*Fouque, J.-P. and H. (2005) A martingale control variate method for option pricing with stochastic volatility.

Numerical Results I: European Options

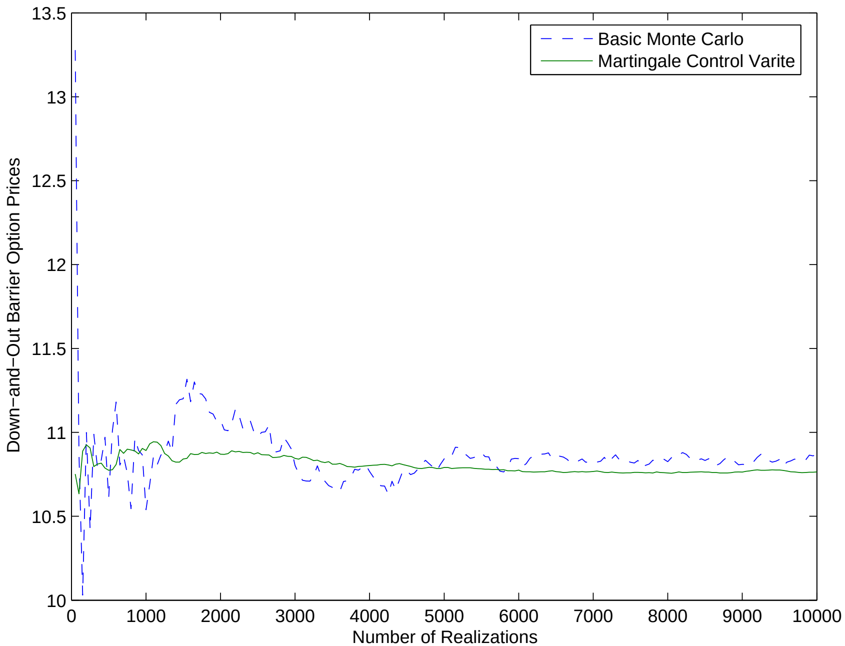
Double Heston Model

$1/\varepsilon$	δ	Std^{BMC}	Std^{MCV}	Ratio
1/75	0.01	0.1103 (7.03)	0.0068 (7.09)	**265**
1/50	0.1	0.1102 (6.97)	0.0073 (7.08)	**230**
1/10	0.5	0.1085 (6.94)	0.0103 (7.03)	**111**
1/5	1	0.1063 (6.91)	0.0113 (6.99)	**89**

Numerical Results II: Barrier Options

$1/\varepsilon$	δ	Std^{BMC}	Std^{MCV}	Ratio
100	0.01	0.2822 (10.82)	0.0304 (10.85)	**86**
75	0.1	0.2047 (10.77)	0.0306 (10.76)	**45**
50	1	0.2455 (11.21)	0.0474 (11.10)	**27**
25	10	0.2604 (12.62)	0.0417 (12.44)	**39**

Variance Reduction: a down and out Barrier Option



American Option Pricing Problem

Given the risk-neutral prob. space $\left(\Omega, \mathcal{F}, I\!P^{\star}, \mathcal{F}_{[0:T]}\right)$, an Americcan option pricing problem is formulated as an optimal stopping problem:

$$P_{am}(0, S_0) = \sup_{0 \le \tau \le T} I\!E^{\star}\left\{e^{-r\tau} H(S_\tau) | \mathcal{F}_0\right\},$$

where τ is any $\mathcal{F}_t$-adapted stopping time and S is the underlying asset price (diffusion) process.

Variational Inequality Characterization

It can be shown that $P_{am}(t,x)$ admits the classical solution of the PDIE

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathcal{A}_{\mathcal{S}}\right) P_{am}(t,x) \leq 0 \\ P_{am}(t,x) \geq H(x) \\ \left(\left(\frac{\partial}{\partial t} + \mathcal{A}_{\mathcal{S}}\right) P_{am}(t,x)\right) \left(P_{am}(t,x) - H(x)\right) = 0. \end{cases}$$

Or one can solve a free boundary problem in PDE.
Deterministic schemes are highly sensitive to dimensionalities.

Recent Development on Efficient Monte Carlo Pricing Algorithms

(1) Primal Approach: approximate optimal stopping rule
Tsitsiklis and Van Roy (2001), Longstaff and Schwartz (2001)

(2) Dual Approach: approximate (super-)martingales
Rogers (2002) used martingale approxim..
Haugh and Kogan (2004) used super-martingale approxim..

Primal Approach (I)
A Low-Biased Solution

For any stopping time $\tilde{\tau}$, a lower solution is deduced

$$\begin{aligned} P_{am}(0, S_0) &= \sup_{0 \le \tau \le T} \mathbb{E}^\star \left\{ e^{-r\tau} H(S_\tau) | \mathcal{F}_0 \right\}, \\ &\ge \mathbb{E}^\star \left\{ e^{-r\tilde{\tau}} H(S_{\tilde{\tau}}) | \mathcal{F}_0 \right\} \equiv P_{am}^{low}(0, S_0) \end{aligned}$$

$\tilde{\tau}$ can be estimated from least squares methods from the dynamic programming formulation.

Primal Approach (II)
Variance Reduction

By martingale control variate methods*,

$$\mathbb{E}^{\star}\left\{e^{-r\tilde{\tau}}H(S_{\tilde{\tau}})\right\} = \mathbb{E}^{\star}\left\{e^{-r\tilde{\tau}}H(S_{\tilde{\tau}}) - \mathcal{M}_{\tilde{\tau}}\right\},$$

where $\mathcal{M}_{\tilde{\tau}} = \int_0^{\tilde{\tau}} e^{-rs}\frac{\partial \tilde{P}}{\partial x}(s, S_s)\sigma_s S_s dW_s$, and $\tilde{P}$ is an approxim. of the option price, one can reduce the variance of the basic Monte Carlo estimator.
Given $\tilde{\tau}$, we transform an American option problem to a **Barrier option problem.**

*Fouque and H. (2006)

Dual Approach (I)
A Direct Monte Carlo Simiulation

Rogers (2002) derived the duality of the American option problem by

$$P_{am}(0, S_0) = \inf_{\mathcal{M} \in H_0^1} \mathbb{E}^\star \left\{ \sup_{0 \le t \le T} \left(e^{-rt} H(S_t) - \mathcal{M}_t \right) \right\},$$

H_0^1={ all integrable martingales but vanish at time 0}

proof:

$\geq$: trivial

$\leq$: use Doob-Meyer decomposition of a supermartingale. (infemum can be attained)

Dual Approach (II)

A High-Biased Solution

Given any martingale $\tilde{M} \in H_0^1$, one deduces an upper solution

$$
\begin{aligned}
P_{am}^{high}(0, S_0) &\equiv I\!\!E^{\star}\left\{ \sup_{0 \le t \le T} \left(e^{-rt} H(S_t) - \tilde{M}_t \right) \right\} \\
&\ge P_{am}(0, S_0)
\end{aligned}
$$

Given $\tilde{M}$ we transform an American option problem to a **Lookback option problem.**

Dual Approach (III)
Error Bound Estimate

Lemma 1: For any given martingale $\tilde{M} \in H_0^1$, $P_{am}^{high}(0, S_0) \leq P_{am}(0, S_0) + \mathbb{E}^\star \left\{ \left| M_T^\star - \tilde{M}_t \right| \right\}$.

Proof:

$$P_{am}^{high}(0, S_0)$$
$$= \mathbb{E}^\star \left\{ \sup_{0 \leq t \leq T} \left(e^{-rt} H(S_t) - M_t^\star + M_t^\star - \tilde{M}_t \right) \right\}$$
$$\leq P_{am}(0, S_0) + \mathbb{E}^\star \left\{ \sup_{0 \leq t \leq T} \left(M_t^\star - \tilde{M}_t \right) \right\}$$
$$\leq P_{am}(0, S_0) + \sqrt{Var \left\{ M_T^\star - \tilde{M}_T \right\}}$$

Variance Bound V.S. Error Bound

Lemma 2: For low-biased solution, the variance of its MCV estimator is

$$Var\left(H(S_{\underline{\tau}}) - \mathcal{M}(\underline{P}; \underline{\tau})\right) = Var\left(\mathcal{M}\left(\mathcal{P} - \underline{P}\right)\right)$$
$$= E\left(\int_0^{\underline{\tau}} e^{-2rs}\left(\frac{\partial P_{am}}{\partial x} - \frac{\partial \underline{P}}{\partial x}\right)\sigma_s^2 S_s^2 ds\right)$$
$$\leq E\left(\int_0^{T} e^{-2rs}\left(\frac{\partial P_{am}}{\partial x} - \frac{\partial \underline{P}}{\partial x}\right)\sigma_s^2 S_s^2 ds\right) \equiv VB$$

For high-biased solution, the error bound was shown

$$P_{am}^{high}(0, S_0) - P_{am}(0, S_0) \leq \sqrt{VB}.$$

Remark: the STD of lower solution and the error bound of higher solution are bounded from above by the same quantity.

Numerical Results III: Low-Biased Solution

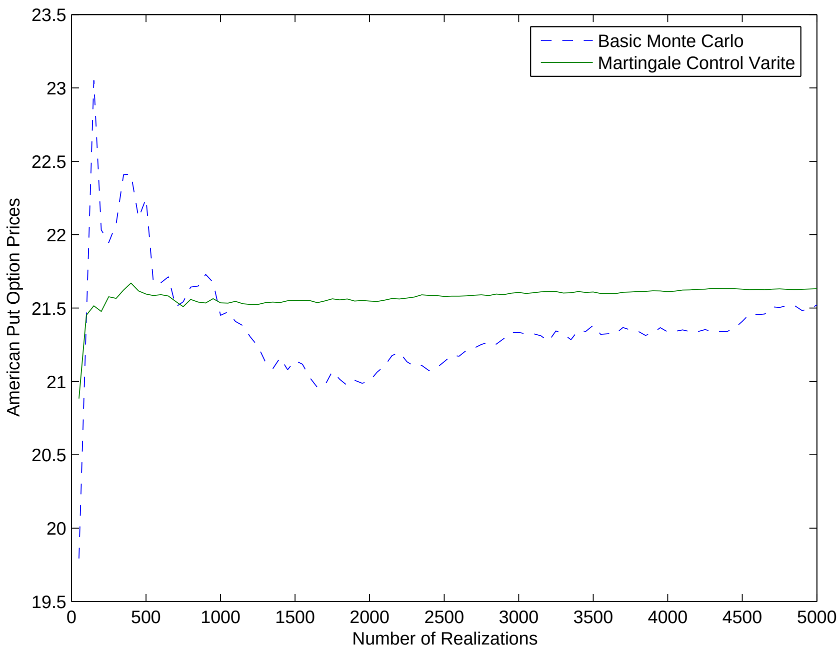
Primal Approach

Use the Least-squares method, which provides a biased **lower** bound solution.

$1/\varepsilon$	δ	$Std_{low}^{Pri_{MC}}$	$Std_{low}^{Pri_{CV}}(P_{BAW})$	Ratio
100	0.01	0.235 (21.43)	0.024 (21.59)	**96**
75	0.1	0.256 (21.48)	0.028 (21.80)	**81**
50	1	0.257 (21.52)	0.035 (21.63)	**54**
25	10	0.260 (21.96)	0.045 (21.32)	**32**

Ref: G. Barone-Adesi and R. E. Whaley, "Efficient Analytic Approximation of American Option Values," The Journal of Finance, Vol. XLII, No. 2, June 1987.

Variance Reduction: American Options



Numerical Results IV: Low-Biased Solution VS High-Biased Solution

Use our control martingale. (Rogers' approach is not easy to generalized to SV models.)

$1/\varepsilon$	δ	$Std_{low}^{Pri_{CV}}$	Std_{high}^{Dul}
100	0.01	0.0240 (21.59)	0.0239 (22.29)
75	0.1	0.0286 (21.80)	0.0271 (22.33)
50	1	0.0350 (21.63)	0.0334 (22.37)
25	10	0.0453 (21.32)	0.0433 (22.29)

Conclusion

- Martingale control variate method is very general for option pricing problems. The control is related to the accumulative value of delta-hedging portfolios.
- This variance analysis technique can also be applied to characterize the **error bound analysis** under randomize **Quasi-MC** methods.
- some future works on importance sampling and the use of statistical estimation..etc.

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