

The Role of New Information in Option Pricing: Estimation issues for the State Space Model

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Problem Statement

- In real world, we have many data analysis involve estimation unknown quantities from some given observations.
- In state-space model, if joins new other to be possible to obtain the information whether can be helpful to its estimation both on unobservable state variables and unknown parameters in the model.

In intuition, we join to have the correlation the recent information are helpful to the model estimate. For instance, in finance model, the REE (Rational Expectations Equilibrium Models) that use the market price and individual private information extrapolated other people's private information to get the equilibrium price. Although this model has in some logical questions, but this also provides everybody a very interesting direction to guess the equilibrium price.

Related literature

- There are two kinds of literature for estimation of the state space model in financial model. One kind is modelling stock price and volatility to financial market. The other kind is involving option price or others derivatives. The first kind model is usually estimated by generalized method of moments (GMM) or Markov Chain Monte Carlo (MCMC) methods.
- Hansen and Scheinkman (1995) show in an important paper how to apply the GMM approach in such situations. They derive moment restrictions in continuous-time models with discretely sampled data. Their paper is quite technical and is difficult to apply when there are unobserved state variables, such as stochastic volatility, for example.

Related literature

- Pan (2002) provide a “implied-state” generalized method of moments (IS-GMM) estimation to solve this problem. They proxy for the unobserved volatility, with an option implied volatility, inverted from the time- t spot price, and a near-the-money short-dated option price, using the model-implied option-pricing formula.
- Recently, MCMC methods have been used in the estimation of continuous time models. Jacquier, Polson, and Rossi (1994) developed this approach to analyze stochastic volatility models. Papers by Eraker (1998) and Jones (1998) are examples of this approach that have recently used this approach in finance.

Related literature

- Kitagawa (1998), Kitagawa and Sato (2000) use the self-organising state model to estimate a stochastic volatility model. They develop a self-organising state model wherein the state and unknown parameters are estimated simultaneously.
- Bhar, Chiarella and Runggaldier (2004) use the forward contracts and option into the state space model. But, their model assume the market price of risk replaced by the volatility. They suggest the Kalman filter to estimate unobservable market price of risk. Runggaldier (2004) extend their results to general setting for partial information.

State Space Model

- The general state space form applies to a multivariate time series, y_t , containing N elements.
- In general the elements of α_t (state variables) are not observable. They are known to be generated by a first-order Markov process, *transition equation*

$$\alpha_t = T_t \alpha_{t-1} + c_t + R_t \eta_t, \quad t = 1, \dots, T$$

where T_t is an $m \times m$ matrix, c_t is an $m \times 1$ vector, R_t is an $m \times g$ matrix and η_t is a $g \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix Q_t ,

$$E(\eta_t) = 0, \quad Var(\eta_t) = Q_t.$$

State Space Model

- The observable variables are related to an $m \times 1$ state vector, α_t , via a *measurement equation*

$$y_t = Z_t \alpha_t + d_t + \varepsilon_t, \quad t = 1, \dots, T$$

where Z_t is an $N \times m$ matrix, d_t is an $N \times 1$ vector and ε_t is an $N \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix H_t ,

$$E(\varepsilon_t) = 0, \quad \text{Var}(\varepsilon_t) = H_t.$$

Example(Heston model (1993))

The process for the stock price S will become

$$\begin{aligned} dS_t &= (r_t - q_t + \eta^S V_t) S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \\ dV_t &= \kappa_v (\bar{v} - V_t) dt + \sigma_v \sqrt{V_t} (\rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)}). \end{aligned} \quad (1)$$

This is a stochastic volatility model. Under Q measure is

$$\begin{aligned} dS_t &= (r_t - q_t) S_t dt + \sqrt{V_t} S_t dW_t^{(1)}(Q) \\ dV_t &= \kappa_v (\bar{v} - V_t + \eta^v V_t) dt + \sigma_v \sqrt{V_t} (\rho dW_t^{(1)}(Q) \\ &\quad + \sqrt{1 - \rho^2} dW_t^{(2)}(Q)). \end{aligned}$$

For simplify, $\rho = 1$, we will use the new notation for stochastic volatility model

$$\begin{aligned}dS_t &= \mu_t S_t dt + \sqrt{V_t} S_t dB_t \\dV_t &= \kappa_v (\bar{v} - V_t) dt + \sigma_v \sqrt{V_t} dB_t\end{aligned}$$

under P measure and

$$\begin{aligned}dS_t &= (r_t - q_t) S_t dt + \sqrt{V_t} S_t dB_t^Q \\dV_t &= \kappa_v^* (\bar{v}^* - V_t) dt + \sigma_v \sqrt{V_t} dB_t^Q\end{aligned}$$

under Q measure, where $dB_t^Q = dB_t + \lambda_t dt$ and λ_t is referred to market price of risk.

Therefore,

$$\sqrt{V_t} S_t \lambda_t = (\mu_t + q_t - r_t) S_t,$$
$$\lambda_t = \frac{\mu_t + q_t - r_t}{\sqrt{V_t}}.$$

Filter Methods

- Kalman Filter
- Particle Filter
(closely related names are bootstrap filters, Monte Carlo filters, et al.)

State Space Model

- If data are modelled by a linear Gaussian state-space model, it is possible to derive an exact analytical expression to compute the evolving sequence of posterior distribution.
- The Kalman filter is a recursive procedure for computing the optimal estimator of the state vector at time t , based on the information available at time t .
- The optimal estimator of α_t in the sense that it minimises the mean square error(MSE).

State Space Model

$$\begin{aligned}
 dS &= \mu S dt + \sqrt{V} S dB \\
 dC &= (r + \lambda \sigma_C) C dt + \sigma_C C dB \\
 dV &= \kappa(\bar{v} - V) dt + b\sqrt{V} dB
 \end{aligned} \tag{2}$$

where state variables $X = [S, C, V]'$ and

$$X_k = a_k + T_k X_{k-1} + R_k \eta_k$$

where

$$a_k = \begin{bmatrix} 0 \\ 0 \\ \kappa \bar{v} \Delta t \end{bmatrix}, \quad T_k = \begin{bmatrix} 1 + (r - q + \lambda_{k-1} \sigma) \Delta t & 0 & 0 \\ 0 & 1 + (r + \lambda_{k-1} \sigma_C) \Delta t & 0 \\ 0 & 0 & 1 - \kappa \Delta t \end{bmatrix}$$

State Space Model

where η is a random variable with $E(\eta_k) = 0$, $Var(\eta_k) = \Delta t$. The observation equation is

$$Y_k = ZX_k + \varepsilon_k$$

where

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where ε is a random variable with $E(\varepsilon_k) = 0$, $Var(\varepsilon_k) = H$. The case without including C is

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Kalman Filter

$$\begin{aligned}
 X_{t|t-1} &= T_t X_{t-1} + a_t \\
 P_{t|t-1} &= T_t P_{t-1} T_t' + R_t Q R_t' \quad t = 1, 2, \dots, n \\
 X_t &= X_{t|t-1} + P_{t|t-1} Z' F_t^{-1} (Y_t - Z X_{t|t-1}) \\
 P_t &= P_{t|t-1} - P_{t|t-1} Z' F_t^{-1} Z P_{t|t-1} \\
 F_t &= Z P_{t|t-1} Z' + H.
 \end{aligned}$$

The log likelihood function is

$$\begin{aligned}
 \log L &= -\frac{mn}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^n \log |F_t| \\
 &\quad - \frac{1}{2} \sum_{t=1}^n (Y_t - Z X_{t|t-1})' F_t^{-1} (Y_t - Z X_{t|t-1}).
 \end{aligned}$$

Particle Filter

- The key idea of the bootstrap filter is to eliminate the particles having low importance weight $\tilde{w}_t^{(i)}$ and to multiply particles having high importance weight (Gordon et al. 1993).
- The unobserved signal (hidden states) $\{x_t; t \in \mathbb{N}\}$ is modelled as a Markov process of initial distribution $p(x_0)$ and transition equation $p(x_t|x_{t-1})$. The observations $\{y_t; t \in \mathbb{N}^*\}$ are assumed to be conditionally independent given the process $\{x_t; t \in \mathbb{N}\}$ and of marginal distribution $p(y_t|x_t)$.

$$\begin{array}{ll}
 p(x_0) & \\
 p(x_t|x_{t-1}) & \text{for } t \geq 1 \\
 p(y_t|x_t) & \text{for } t \geq 1
 \end{array}$$

Denote $x_{0:t} \equiv \{x_0, \dots, x_t\}$ and $y_{1:t} \equiv \{y_1, \dots, y_t\}$.

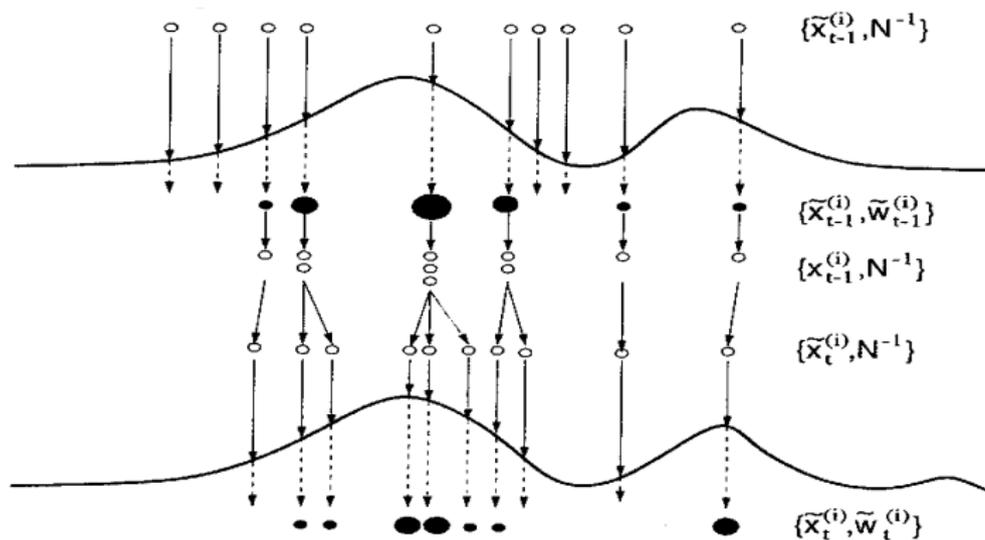
Particle Filter

- 1 Initialisation, $t=0$.
 - For $i = 1, \dots, N$, sample $x_0^{(i)} \sim p(x_0)$ and set $t = 1$.
- 2 Importance sampling step
 - For $i = 1, \dots, N$, sample $\tilde{x}_t^{(i)} \sim p(x_t | x_{t-1}^{(i)})$ and set $\tilde{x}_{0:t}^{(i)} = (\tilde{x}_{0:t-1}^{(i)}, \tilde{x}_t^{(i)})$.
 - For $i = 1, \dots, N$, evaluate the importance weights

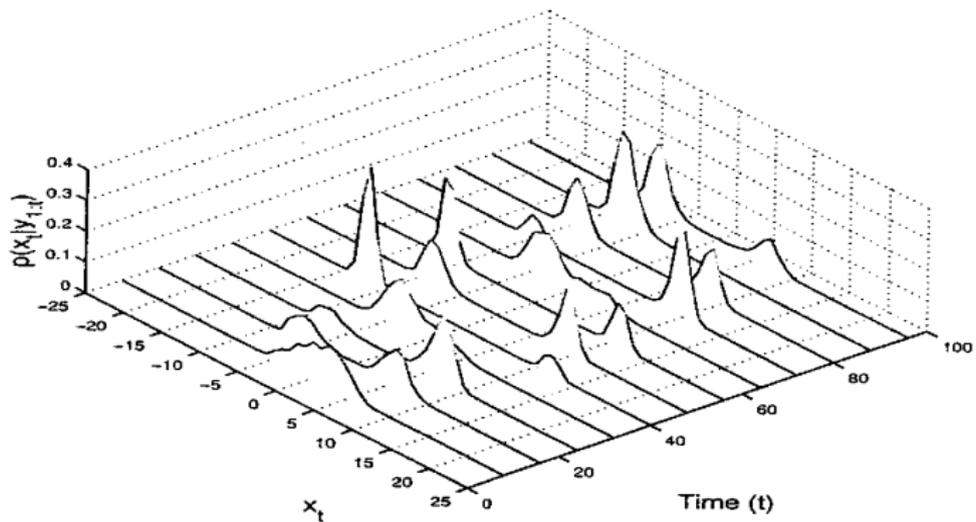
$$\tilde{w}_t^{(i)} = p(y_t | \tilde{x}_t^{(i)}).$$

- Normalise weights.
- 3 Selection step
 - Resample with replacement N particles $(x_{0:t}^{(i)}; i = 1, \dots, N)$ from set $(\tilde{x}_{0:t}^{(i)}; i = 1, \dots, N)$ by importance weights.
 - Set $t \leftarrow t + 1$ and go to step 2.

Particle Filter



Particle Filter



Particle Filter

Easiest way to approximate the likelihood function at fixed θ from

$$\begin{aligned}
 L(\theta|y_{1:T}) &= p(y_{1:T}|\theta) = \prod_{t=1}^T p(y_t|y_{1:t-1}, \theta) \\
 &= \prod_{t=1}^T \int p(y_t|x_t, \eta)p(x_t|y_{1:t-1}, \theta)dx_t \\
 &= \prod_{t=1}^T E[p(y_t|x_t, \eta)|y_{1:t-1}, \theta] \\
 &\approx \prod_{t=1}^T \frac{1}{N} \sum_{j=1}^N p(y_t|x_{pr,t}^{(j)}, \eta)
 \end{aligned}$$

where $x_{pr,t}^{(j)} \sim p(x_t|x_{t-1}^{(j)}, \theta)dx_t$. Approximate

$$\log L(\theta|y_{1:T}) + T \log N \approx \sum_{t=1}^T \log\left(\sum_{j=1}^N p(y_t|x_{pr,t}^{(j)}, \eta)\right).$$

We compare the estimation of state variable volatility under controlling single parameter changed. The Monte Carlo size is 10000. If

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0.09 \end{pmatrix}.$$

Kalman Filter

| | \bar{v} is known | | \bar{v} is unknown | | | |
|-----------------|--------------------|------------------|----------------------|-----------------------|------------------|-----------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{v} - \bar{v} $ | state variable | $ \hat{v} - \bar{v} $ |
| $\bar{v} = 0.1$ | 0.014 (0.022) | 0.012 (0.016) | 0.027 (0.037) | 0.012 (0.018) | 0.025 (0.034) | 0.009 (0.015) |
| $\bar{v} = 0.3$ | 0.031 (0.049) | 0.028 (0.041) | 0.046 (0.056) | 0.021 (0.024) | 0.044 (0.054) | 0.020 (0.022) |
| $\bar{v} = 0.5$ | 0.038 (0.051) | 0.034 (0.048) | 0.058 (0.066) | 0.033 (0.046) | 0.053 (0.061) | 0.031 (0.039) |

Table: The difference of state variable V and \hat{v} for included C or no included C case with parameters $r = 0.05$, $q = 0.01$, $\kappa = 5$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Practical Filter

| | \bar{v} is known | | \bar{v} is unknown | | | |
|-----------------|--------------------|------------------|----------------------|-----------------------|------------------|-----------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{v} - \bar{v} $ | state variable | $ \hat{v} - \bar{v} $ |
| $\bar{v} = 0.1$ | 0.014 (0.021) | 0.011 (0.013) | 0.026 (0.035) | 0.012 (0.019) | 0.022 (0.031) | 0.007 (0.012) |
| $\bar{v} = 0.3$ | 0.030 (0.044) | 0.025 (0.038) | 0.044 (0.053) | 0.020 (0.022) | 0.040 (0.048) | 0.016 (0.019) |
| $\bar{v} = 0.5$ | 0.037 (0.049) | 0.031 (0.044) | 0.057 (0.061) | 0.031 (0.043) | 0.051 (0.056) | 0.029 (0.038) |

Table: The difference of state variable V and \hat{v} for included C or no included C case with parameters $N = 100$, $r = 0.05$, $q = 0.01$, $\kappa = 5$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Kalman Filter

| | κ is known | | κ is unknown | | | |
|---------------|-------------------|------------------|---------------------|---------------------------|------------------|---------------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{\kappa} - \kappa $ | state variable | $ \hat{\kappa} - \kappa $ |
| $\kappa = 5$ | 0.012 (0.016) | 0.011 (0.014) | 0.038 (0.045) | 0.852 (0.472) | 0.034 (0.041) | 0.837 (0.460) |
| $\kappa = 10$ | 0.028 (0.037) | 0.025 (0.035) | 0.061 (0.108) | 1.241 (0.806) | 0.058 (0.104) | 1.127 (0.798) |
| $\kappa = 20$ | 0.032 (0.055) | 0.031 (0.053) | 0.152 (0.241) | 2.139 (1.493) | 0.146 (0.235) | 2.063 (1.469) |

Table: The difference of state variable V and $\hat{\kappa}$ for included C or no included C case with parameters $r = 0.05$, $q = 0.01$, $\bar{v} = 0.1$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Practical Filter

| | κ is known | | κ is unknown | | | |
|---------------|-------------------|------------------|---------------------|---------------------------|------------------|---------------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{\kappa} - \kappa $ | state variable | $ \hat{\kappa} - \kappa $ |
| $\kappa = 5$ | 0.012 (0.014) | 0.010 (0.011) | 0.037 (0.043) | 0.838 (0.442) | 0.029 (0.034) | 0.773 (0.372) |
| $\kappa = 10$ | 0.026 (0.034) | 0.022 (0.031) | 0.059 (0.101) | 1.151 (0.726) | 0.051 (0.092) | 1.062 (0.685) |
| $\kappa = 20$ | 0.031 (0.052) | 0.026 (0.047) | 0.147 (0.225) | 1.967 (1.235) | 0.124 (0.183) | 1.847 (1.108) |

Table: The difference of state variable V and $\hat{\kappa}$ for included C or no included C case with parameters $N = 100$, $r = 0.05$, $q = 0.01$, $\bar{v} = 0.1$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Kalman Filter

| | σ is known | | σ is unknown | | | |
|----------------|-------------------|------------------|---------------------|---------------------------|------------------|---------------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{\sigma} - \sigma $ | state variable | $ \hat{\sigma} - \sigma $ |
| $\sigma = 0.1$ | 0.023 (0.028) | 0.021 (0.024) | 0.044 (0.053) | 0.037 (0.034) | 0.040 (0.047) | 0.035 (0.031) |
| $\sigma = 0.3$ | 0.047 (0.056) | 0.045 (0.047) | 0.072 (0.084) | 0.131 (0.075) | 0.068 (0.081) | 0.123 (0.068) |
| $\sigma = 0.5$ | 0.061 (0.059) | 0.057 (0.051) | 0.114 (0.108) | 0.183 (0.093) | 0.107 (0.103) | 0.177 (0.086) |

Table: The difference of state variable V and $\hat{\sigma}$ for included C or no included C case with parameters $r = 0.05$, $q = 0.01$, $\bar{v} = 0.1$, $\kappa = 5$, $n = 50$, $\Delta t = 1/250$.

Practical Filter

| | σ is known | | σ is unknown | | | |
|----------------|-------------------|------------------|---------------------|---------------------------|------------------|---------------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{\sigma} - \sigma $ | state variable | $ \hat{\sigma} - \sigma $ |
| $\sigma = 0.1$ | 0.021 (0.026) | 0.020 (0.023) | 0.043 (0.051) | 0.035 (0.033) | 0.035 (0.042) | 0.027 (0.024) |
| $\sigma = 0.3$ | 0.043 (0.047) | 0.039 (0.043) | 0.068 (0.077) | 0.128 (0.073) | 0.052 (0.067) | 0.073 (0.045) |
| $\sigma = 0.5$ | 0.058 (0.055) | 0.053 (0.046) | 0.109 (0.105) | 0.179 (0.091) | 0.088 (0.072) | 0.136 (0.065) |

Table: The difference of state variable V and $\hat{\sigma}$ for included C or no included C case with parameters $N = 100$, $r = 0.05$, $q = 0.01$, $\bar{v} = 0.1$, $\kappa = 5$, $n = 50$, $\Delta t = 1/250$.

If

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Kalman Filter

| | \bar{v} is known | | \bar{v} is unknown | | | |
|-----------------|--------------------|------------------|----------------------|-----------------------|------------------|-----------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{v} - \bar{v} $ | state variable | $ \hat{v} - \bar{v} $ |
| $\bar{v} = 0.1$ | 0.014 (0.022) | 0.016 (0.024) | 0.027 (0.037) | 0.012 (0.018) | 0.029 (0.041) | 0.009 (0.015) |
| $\bar{v} = 0.3$ | 0.031 (0.049) | 0.037 (0.042) | 0.046 (0.056) | 0.021 (0.024) | 0.054 (0.064) | 0.024 (0.028) |
| $\bar{v} = 0.5$ | 0.038 (0.051) | 0.044 (0.058) | 0.058 (0.066) | 0.033 (0.046) | 0.079 (0.071) | 0.063 (0.059) |

Table: The difference of state variable V and \hat{v} for included C or no included C case with parameters $r = 0.05$, $q = 0.01$, $\kappa = 5$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Practical Filter

| | \bar{v} is known | | \bar{v} is unknown | | | |
|-----------------|--------------------|------------------|----------------------|-----------------------|------------------|-----------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{v} - \bar{v} $ | state variable | $ \hat{v} - \bar{v} $ |
| $\bar{v} = 0.1$ | 0.014 (0.021) | 0.013 (0.021) | 0.026 (0.035) | 0.012 (0.019) | 0.025 (0.033) | 0.008 (0.015) |
| $\bar{v} = 0.3$ | 0.030 (0.044) | 0.029 (0.041) | 0.044 (0.053) | 0.020 (0.022) | 0.043 (0.049) | 0.019 (0.028) |
| $\bar{v} = 0.5$ | 0.037 (0.049) | 0.032 (0.043) | 0.057 (0.061) | 0.031 (0.043) | 0.054 (0.058) | 0.030 (0.039) |

Table: The difference of state variable V and \hat{v} for included C or no included C case with parameters $N = 100$, $r = 0.05$, $q = 0.01$, $\kappa = 5$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

If $\rho = \text{Corr}(B_1, B_2) = 0.3$,

| | \bar{v} is known | | \bar{v} is unknown | | | |
|-----------------|--------------------|------------------|----------------------|-----------------------|------------------|-----------------------|
| | no include C | include C | no include C | | include C | |
| | state variable | state variable | state variable | $ \hat{v} - \bar{v} $ | state variable | $ \hat{v} - \bar{v} $ |
| $\bar{v} = 0.1$ | 0.301 (0.217) | 0.312 (0.204) | 0.420 (0.337) | 0.082 (0.091) | 0.411 (0.342) | 0.077 (0.086) |
| $\bar{v} = 0.3$ | 0.833 (0.741) | 0.852 (0.785) | 1.031 (0.856) | 0.213 (0.182) | 1.038 (0.874) | 0.234 (0.248) |
| $\bar{v} = 0.5$ | 2.331 (1.804) | 2.298 (1.716) | 3.352 (1.582) | 0.425 (0.366) | 3.391 (1.616) | 0.416 (0.379) |

Table: The difference of state variable V and \hat{v} for included C or no included C case with parameters $r = 0.05$, $q = 0.01$, $\kappa = 5$, $\sigma = 0.1$, $n = 50$, $\Delta t = 1/250$.

Out-of-sample forecast

From Findley(2005), assume we are interested in h -step forecasting of time series Y_t , $1 \leq t \leq n$. The out-of-sample h -step forecast of Y_{t+h} is defined by $Y_{t+h|t}$. Define the forecast error by $e_{t+h|t} = Y_{t+h} - Y_{t+h|t}$. The sequence of accumulating sums of squared out-of-sample forecast errors is

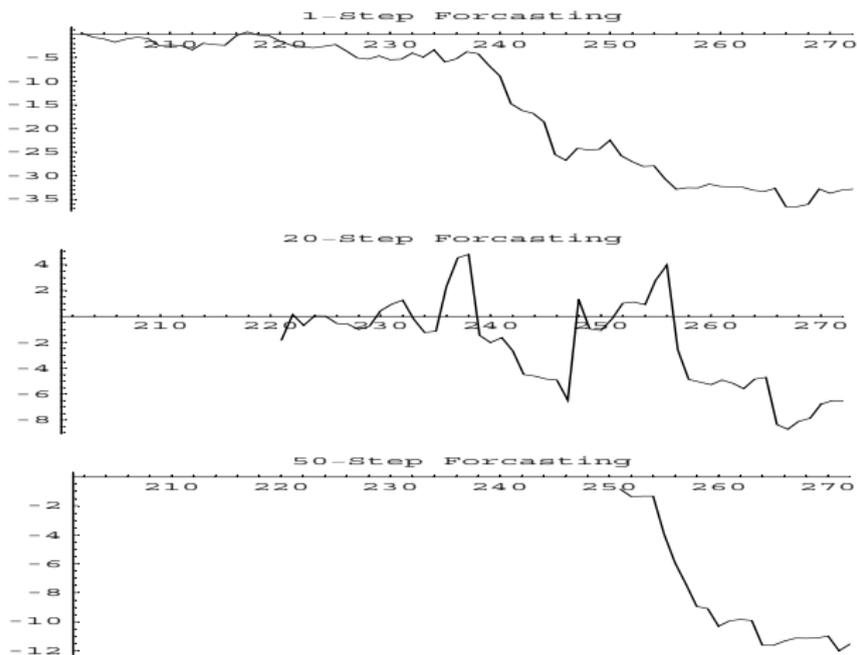
$$SS_{h,M} = \sum_{t=n_0}^M e_{t+h|t}^2, \quad M = n_0, \dots, n - h.$$

Out-of-sample forecast

In Figure, we plot a standardized version of the differences $SS_{h,M}^{(1)} - SS_{h,M}^{(2)}$ defined by

$$SS_{h,M}^{1,2} = \frac{SS_{h,M}^{(1)} - SS_{h,M}^{(2)}}{SS_{h,n-h}^{(2)}/n - h - n_0}, \quad M = n_0, \dots, n - h.$$

Out-of-sample forecast



Summary

Remark: From the Tables, we can observe some interest phenomenon about state variable estimation and parameter estimation. We summary the results as follows:

- The correlation between new information and state variable is very important for estimation. More higher correlation, the estimation will be improved more.
- The quality of new information is a key for state variable estimation. Here, we say that the quality is high, it means smaller the measurement error than original data.

- The state variable estimation is more sensitive than the parameter estimation. Because the parameter estimation can be improved by including the other new information.
- The non-linearity problem will reduce the new information effect, especially in unknown state estimation.

We conclude the observations that the useful new information is both high correlation and high quality for original data.