Risk Management for Linear and Non-Linear Assets- A Bootstrap Method with Importance Resampling to Evaluate Value-at-Risk

reporter: Wang, Ren-Her

July 29, 2006

joint work with Professor Fuh, Cheng-Der and Lin, Shih-Kuei

Outline

1. Value-at-Risk (VaR)
2. Literature
3. Model
4. Linear Portfolio
   - Risk factors with a multivariate normal distribution
   - Risk factors with a multivariate t distribution
5. Nonlinear portfolio
   - Risk factors with a multivariate t distribution
6. Simulation results

- VaR is the loss in market value over the time horizon t that is exceeded with probability $1-p$.
- The risk factors are computed from a univariate **normal distribution** or a multivariate normal distribution (cf. Jorion, 2000, Duffie and Pan, 1997, and J.P. Morgan, 1995).
- There is much empirical evidence suggesting that risk factors, such as log-returns of US stocks, do **not** follow a normal distribution (cf. Blattberg and Gonedes, 1974, Cont, 2001, Fama, 1965, Mandelbrot, 1963, and Rachev and Mittnik, 2000).
- These **heavy tails** are particularly troublesome because VaR attempts to capture the behavior of the portfolio return in the tail.

- Glasserman, Heidelberger and Shahabuddin (2000) use several variance reduction techniques for estimating VaR, such as importance sampling, stratified sampling.
- Glasserman, Heidelberger and Shahabuddin (2002) also discuss the portfolio VaR with heavy-tailed risk factors.

Variance reduction techniques

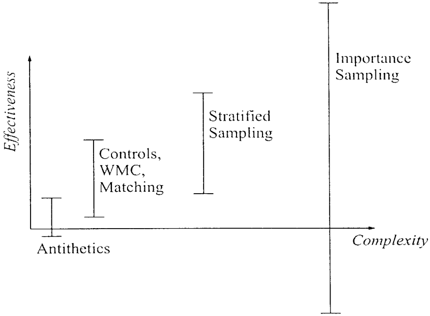

Fig. 4.16. Schematic comparison of some variance reduction techniques

Simulation process

1. Parameter estimation for real data.
2. Parameter bootstrap method for new sampling.
3. Importance sampling for quantile estimation.

Model

- Suppose the portfolio $V(t, \widetilde{S}(t))$ denotes the function of risk factor $S(t)$ and time t, where $\widetilde{S}(t) = (S_1(t), \ldots, S_n(t))'$ is the n underlying assets of the portfolio at time t, and the value of the portfolio at time $t+1$ is $V(t+1, \widetilde{S}(t+1))$. The loss in portfolio value during the holding period is $L = -\Delta V$ where $\Delta V = V(t+1, \widetilde{S}(t+1)) - V(t, \widetilde{S}(t))$, and the VaR, l_p, associated with a given probability p is defined by

$$P(L > l_p) = p. \tag{1}$$

- Assume the density of the portfolio return is symmetric. We can change the density of the portfolio loss into the density of the portfolio return,

$$P(R(t) > r_p) = p, \tag{2}$$

where $r_p = 2\mu - l_p/V$, and μ denotes the mean of the portfolio.

Linear Portfolio

- Let $\tilde{w}(t) = (w_1(t), \ldots, w_n(t))'$ denote a portfolio weight vector of the investment assets for the portfolio value where $w_i(t)$ is an adapted process, i.e., $\mathcal{F}_t$-measurable, and $\tilde{r}(t) = (r_1(t), \ldots, r_n(t))'$ is a vector of the discrete return of the assets, where $r_i(t) = (S_i(t+1) - S_i(t))/S_i(t)$. Then the return of the portfolio at time t is the linear combination of the asset returns multiplied by the portfolio weight vector, denoted as

$$R(t) = \tilde{w}'(t)\tilde{r}(t) \tag{3}$$

Parameter estimation

(1) The observed data $\tilde{r}^F$ is assumed to come from distribution F, the unknown parameters of the distribution are estimated by the maximum likelihood estimate or the method of moment. Then, we obtain the empirical distribution $\widehat{F}$ by the plug in principle.

(2) Select B independent bootstrap samples $\tilde{r}^{F,1}, \tilde{r}^{F,2}, \ldots, \tilde{r}^{F,B}$, each consisting of T data values drawn with replacement from $\hat{F}$.

(3) Evaluate the bootstrap replication corresponding to each bootstrap sample,

$$\hat{p}_b^F = \frac{1}{T}\sum_{t=1}^{T} 1_{A^F(b,t)} \qquad b = 1, 2, \cdots, B, \tag{4}$$

where $A^F(b,t) = \{\tilde{r}^F(b,t) : f(R^F(b,t)) = R^F(b,t) - r_p^F = \tilde{w}'(t)\tilde{r}^F(b,t) - r_p^F > 0\}$.

(4) Estimate the probability p and standard error $se_p(\hat{p})$ by using the sample standard deviation of B replications.

Set $r_i^N(t) = \Delta S_i(t)/S_i(t)$, $i = 1, 2, \ldots, n$, then the vector of the asset returns follows

$$\tilde{r}^N(t) = \begin{bmatrix} r_1^N(t) \\ \vdots \\ r_n^N(t) \end{bmatrix} = \begin{bmatrix} \mu_1 + \sigma_1\varepsilon_1 \\ \vdots \\ \mu_n + \sigma_n\varepsilon_n \end{bmatrix} = \tilde{\mu} + \sigma\tilde{\varepsilon}. \tag{5}$$

Our basic approach is to use model to approximate the portfolio return loss probability, and then apply it to obtain importance resampling distribution for **variance reduction**.

We are interested in the event $A^N(t) = \{\tilde{r}^N(t) : f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N > 0\}$. Note that we can rewrite

$$f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N = \tilde{\sigma}_w'\tilde{\varepsilon} + \tilde{w}'(t)\tilde{\mu} - r_p^N \quad (6)$$

$$\stackrel{d}{=} KZ + \tilde{w}'(t)\tilde{\mu} - r_p^N, \quad (7)$$

where $\tilde{\sigma}_w = \tilde{w}'(t)\sigma = (w_1(t)\sigma_1, w_2(t)\sigma_2, \ldots, w_n(t)\sigma_n)'$, $K = \sqrt{\tilde{\sigma}_w'\Sigma\tilde{\sigma}_w}$, Z is a normal distribution with mean 0 and variance 1, $r_p^N (r_p^N = Kz_p + \tilde{w}'(t)\tilde{\mu})$ is the quantile of the portfolio return with a multivariate normal assumption, $\stackrel{d}{=}$ means equal in distribution, and z_p is the quantile of the standard normal density.

By using the Cholesky decomposition for Σ, we have

$$f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N \stackrel{d}{=} \tilde{\sigma}_w' C \widetilde{Z} + \tilde{w}'(t)\tilde{\mu} - r_p^N \quad (8)$$

$$\stackrel{d}{=} DZ + \tilde{w}'(t)\tilde{\mu} - r_p^N, \quad (9)$$

where $C = [c_{ij}]$ is used by Cholesky decomposition for Σ, and $D = \sqrt{\sum_{j=1}^n (\sum_{i=1}^n w_i(t)\sigma_i c_{ij})^2} = K$.

Glasserman *et al.* (1999a) describes how to select the tilting measure in risk factors. Standard exponential embedding leads to

$$\frac{d\mathbb{P}^{\theta}_{\tilde{a}^N}}{d\mathbb{P}} = \exp\{\theta f(R^N(t)) - \psi^N(\theta)\}, \tag{10}$$

where $d\mathbb{P}$ is the original probability measure, and $d\mathbb{P}^{\theta}_{\tilde{a}^N(\theta)}$ is the tilting measure from the multivariate normal distribution $MN(\tilde{0}, I)$ to the multivariate normal distribution $MN(\tilde{a}^N(\theta), I)$. Here

$$\psi^N(\theta) = \log E(\exp\{\theta f(R^N(t))\}) = \theta(\tilde{w}'(t)\tilde{\mu} - r_p^N) + \frac{1}{2}\theta^2 K^2. \tag{11}$$

Let $A_\theta^N(t) = \{\tilde{r}_\theta^N(t) : f(R_\theta^N(t)) = \tilde{w}'(t)\tilde{r}_\theta^N(t) - r_p^N > 0\}$ be the event to be simulated.

Denote

$$\hat{p}^N(\theta) = 1_{A_\theta^N(t)} \exp\{-\theta f(R_\theta^N(t)) + \psi^N(\theta)\}, \tag{12}$$

and let $\tilde{r}_\theta^N(t)$ be drawn from the tilting measure $\mathbb{P}^\theta_{\tilde{a}^N(\theta)}$, then the estimator $1_{A^N(t)}$ is unbiased. That is,

$$E(1_{A^N(t)}) = E^\theta(1_{A_\theta^N(t)} \exp\{-\theta f(R_\theta^N(t)) + \psi^N(\theta)\}) = E^\theta(\hat{p}^N(\theta)) = p.$$

Therefore, we only compute the second moment of the estimator for the tail probability

$$
\begin{aligned}
M_2^N(\theta) &= E^{\theta}(1_{A_\theta^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R_\theta^N(t))\}) \\
&= \exp\{\psi^N(\theta) - \theta \tilde{w}'(t)\tilde{\mu} + \theta r_p^N + \frac{1}{2}\theta^2 K^2\} \int_{\lambda_1}^{\infty} \phi(z) dz, \qquad (13)
\end{aligned}
$$

where $\lambda_1 = \theta K + (r_p^N - \tilde{w}'(t)\tilde{\mu})/K$, and $\phi(z)$ is the standard normal density.

Since it is difficult to find the value of θ by minimizing $M_2^N(\theta)$, we will minimize its upper bound (cf. Glasserman *et al.*, 1999a) as follows,

$$\begin{aligned} M_2^N(\theta) &= E^\theta(1_{A_\theta^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R_\theta^N(t))\}) \\ &\leq \exp\{\psi^N(\theta)\}, \end{aligned} \quad (14)$$

because $\exp\{-\theta f(R_\theta^N(t))\} \leq 1$ ($\because f(\tilde{r}_\theta^N(t)) > 0$ and $\theta > 0$), and $1_{A_\theta^N(t)} \leq 1$. Taking log of the bound equation (14) and differentiating θ we have

$$\theta_g^N = \frac{r_p^N - \tilde{w}'(t)\tilde{\mu}}{K^2}. \quad (15)$$

We can also find the new upper bound of the second moment for the estimate of the tail probability by using the inequality (Durrett, 1995), the Laplace method,

$$\int_{\lambda}^{\infty} \exp\{-z^2\}dz \leq \lambda^{-1} \exp\{-\lambda^2/2\}. \tag{16}$$

Compute the new second moment upper bound,

$$\begin{aligned} M_2^N(\theta) &= E^{\theta}(1_{A_{\theta}^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R(t))\}) \\ &\leq \exp\{\psi^N(\theta) - \frac{1}{2}(\frac{r_p^N - \tilde{w}'(t)\tilde{\mu}}{K})^2\} \frac{1}{\sqrt{2\pi}} \frac{K}{r_p^N - \tilde{w}'(t)\tilde{\mu} + \theta K^2} \end{aligned} \tag{17}$$

Then, taking the new bound equation (17) into log and differentiating θ, we obtain the solution of θ_l^N for the second moment of tail probability by the inequality (Durrect, 1995) in the multivariate normal distribution,

$$\theta_l^N = \sqrt{\frac{K^2 + (r_p^N - \tilde{w}'(t)\tilde{\mu})^2}{K^4}}. \quad (18)$$

Risk factors with a multivariate t distribution

- Although the multivariate normal distribution assumption is commonly used in the literature, many empirical studies suggest that the distribution has heavy tails. One of the most pervasive features observed across equity, foreign exchange, and interest rate markets is that they have kurtosis excess, so the distribution of the asset has leptokurtic features.
- The similar process compute the tilting point θ.

- $$\theta_g^T = \frac{r_p^T - \tilde{w}'(t)\tilde{\mu}}{K^2}, \quad (19)$$

  and

  $$\theta_l^T = \frac{(r_p^T - \tilde{w}'(t)\tilde{\mu}) + \sqrt{(r_p^T - \tilde{w}'(t)\tilde{\mu})^2(\nu^2+\nu+1) + K^2\nu^2}}{(\nu+1)K^2}. \quad (20)$$

- Compared with (15), (19) is the tilting formula for the multivariate t distribution. The formulations of (15) and (19) are quite similar, but with different r_p^i, $i = N, T$. Note that as ν tends to infinity, r_p^T converges to r_p^N and the tilting points are the same. By using the same argument, it is easy to see that (18) will converge to (20), as ν tends to infinity.

Quadratic Approximation

- By the delta-gamma approximation (quadratic approximation), the change in portfolio value for the nonlinear portfolio can be written as

$$V(t+1,\tilde{S}(t+1)) - V(t,\tilde{S}(t)) \approx \frac{\partial V}{\partial t}\Delta t + \delta'\Delta\tilde{S}(t) + \frac{1}{2}\Delta\tilde{S}(t)'\Gamma\Delta\tilde{S}(t),$$

where $\frac{\partial V}{\partial t}$ is the change of the portfolio from t to $t+1$, $\delta_i = \frac{\partial V}{\partial S_i}$ denotes the delta approximation of the portfolio for the asset i, $\delta' = [\delta_1, \ldots, \delta_n]$ is the vector of the delta approximation, $\Gamma_{ij} = \frac{\partial^2}{\partial S_i \partial S_j}$ is the gamma approximation of the portfolio for the asset i and asset j, Γ is the matrix of the gamma approximation, and $\Delta\tilde{S}(t)' = [\Delta S_1(t), \ldots, \Delta S_n(t)]$ denotes the change of the assets.

- The loss in portfolio, L, can be rewritten as

$$\begin{aligned} L &\approx a_0 + a'\Delta\tilde{S}(t) + (\Delta\tilde{S}(t))'A\Delta\tilde{S}(t) \\ &= a_0 + a_1'\tilde{r} + \tilde{r}'A_1\tilde{r}, \end{aligned}$$

where $a_0 = \frac{\partial V}{\partial t}\Delta t$ is a scalar, $a = -\delta$, $A = -\frac{1}{2}\Gamma$ is a $m \times m$ matrix, $a_1' = [-S_1\delta_1, \ldots, -S_n\delta_n]$, $(A_1)_{ij} = -\frac{1}{2}\Gamma_{ij}S_iS_j$, and $\tilde{r}' = [\frac{\Delta S_1(t)}{S_1(t)}, \ldots, \frac{\Delta S_n(t)}{S_n(t)}]$ denotes the vector of the discrete returns in the assets.

Risk factors with a multivariate t distribution

Assume the return of the assets is equal to the mean $\tilde{u}$, and a multivariate t distribution with the degree of freedom v as follows:

$$\tilde{r} = \tilde{u} + t_v,$$

where $t_v = \frac{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)}{\sqrt{Y/v}} = \frac{\tilde{\varepsilon}}{\sqrt{Y/v}}$ has a multivariate t distribution with the degree of freedom v, $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ denotes a multivariate normal distribution with zero mean vector and covariance matrix Σ.

Let $CC' = \Sigma$, and $C'A_1C = \Lambda$. Then

$$\begin{aligned} L &= a_0 + a_1'\tilde{r} + \tilde{r}'A_1\tilde{r} \\ &= b_0 + a_1't_v + (t_v)'A_1(t_v) \end{aligned}$$

where $b_0 = a_0 + a_1'\tilde{u} + \tilde{u}'A_1\tilde{u}$. Thus, let $Q \equiv L - b_0$, then

$$\begin{aligned} Q &= a_1't_v + (t_v)'A_1(t_v) \overset{d}{=} a_1'CX + X'C'A_1CX \\ &= b'X + X'\Lambda X = \sum_{j=1}^{n} b_jX_j + \lambda_jX_j^2 \end{aligned} \tag{21}$$

where $b' = a_1'C$. Let $t_v \overset{d}{=} CX = C\frac{\tilde{Z}}{\sqrt{Y/v}}$, where $\tilde{Z}$ has a multivariate normal distribution with zero mean vector and identity covariance matrix I, and

$$X = (X_1, \ldots, X_n) = \frac{\tilde{Z}}{\sqrt{Y/v}},$$

The probability of the loss can be rewritten

$$\begin{aligned} P(L > l_{1-p}) = p &= P(b_0 + Q > l_{1-p}) \\ &= P(\tfrac{Y}{v}(Q - x) > 0) \\ &= P(Q_x > 0), \end{aligned}$$

where $x = l_{1-p} - b_0$ and $Q_x = \frac{Y}{v}(Q - x)$.
Then, use the exponential change of measure (Glasserman et al., 2000 and 2002)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\{\theta Q_x - \psi(\theta)\}, \tag{22}$$

where $\psi(\theta) = \log E(\exp(\theta Q_x))$.

In Glasserman et al. (2002), they obtain the moment generating function of Q_x as follows:

$$\phi(\theta) = \left(1 + \frac{2\theta x}{v} - \sum_{j=1}^{n} \frac{\theta^2 b_j^2}{(1-2\theta\lambda_j)v}\right)^{-\frac{v}{2}} \prod_{j=1}^{n} \frac{1}{\sqrt{1-2\theta\lambda_j}}, \quad (23)$$

and

$$\begin{aligned}\psi(\theta) &= \log\phi(\theta) = -\tfrac{v}{2}\log\left(1 + \tfrac{2\theta x}{v} - \textstyle\sum_{j=1}^{n} \tfrac{\theta^2 b_j^2}{(1-2\theta\lambda_j)v}\right) \\ &\quad + \textstyle\sum_{j=1}^{n} \log \tfrac{1}{\sqrt{1-2\theta\lambda_j}}.\end{aligned} \quad (24)$$

Let $A_\theta(t) = \{\tilde{r}_\theta(t) : Q_x > 0\}$ be the event of interest to be simulated for the tail probability, and let the estimator of the tail probability be

$$\hat{p} = e^{-\theta Q_x + \psi(\theta)} 1_{\{Q_x > 0\}}.$$

The estimator $\hat{p}$ is unbiased in the sense that

$$\begin{aligned} E_\theta(\hat{p}) &= E(1_{\{Q_x > 0\}}) \\ &= P(Q_x > 0) = p. \end{aligned}$$

- We use the Laplace method to find the new upper bound of the second moment.
-

$$\begin{aligned}
E_\theta(\hat{p}^2) \quad & \leq \quad \int_{-\infty}^{\infty} \exp\left\{ \frac{yx\theta}{v} + \psi(\theta) + \frac{b_1^2\theta^2 y}{2(1+2\lambda_1\theta)v} \right\} \frac{1}{\sqrt{1+2\lambda_1\theta}} \left[\frac{1}{\sqrt{2\pi}(h_1(\theta)\sqrt{\frac{y}{v}})} \right. \\
& \quad \left. \exp\left\{ -\frac{(h_1(\theta))^2 \frac{y}{v}}{2} \right\} + \frac{1}{\sqrt{2\pi}(h_2(\theta)\sqrt{\frac{y}{v}})} \exp\left\{ -\frac{(h_1(\theta))^2 \frac{y}{v}}{2} \right\} \right] \frac{1}{\Gamma(\frac{v}{2})\beta^{v/2}} y^{\frac{v}{2}-1} \exp\left\{ - \right. \\
& = \quad \frac{\sqrt{v}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{v}{2})\beta^{v/2}} y^{(\frac{v-1}{2}-1)} \exp\left\{ -(\frac{1}{2} + \frac{h_1(\theta)^2}{2v} - \frac{\theta x}{v} - \frac{b_1^2\theta^2}{2(1+2\lambda_1\theta} \right. \\
& + \quad \frac{\sqrt{v}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{v}{2})\beta^{v/2}} y^{(\frac{v-1}{2}-1)} \exp\left\{ -(\frac{1}{2} + \frac{h_2(\theta)^2}{2v} - \frac{\theta x}{v} - \frac{b_1^2\theta^2}{2(1+2\lambda_1\theta} \right. \\
& = \quad I_1(\theta) + I_2(\theta),
\end{aligned}$$

where

$$
\begin{aligned}
I_1(\theta) &= \frac{\sqrt{v}\,\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)} \\
&\quad \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{v}{2})\beta^{v/2}} y^{(\frac{v-1}{2}-1)} \exp\left\{-(\frac{1}{2}+\frac{h_1(\theta)^2}{2v}-\frac{\theta x}{v}-\frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)v})y\right\} dy, \\
I_2(\theta) &= \frac{\sqrt{v}\,\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)} \\
&\quad \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{v}{2})\beta^{v/2}} y^{(\frac{v-1}{2}-1)} \exp\left\{-(\frac{1}{2}+\frac{h_2(\theta)^2}{2v}-\frac{\theta x}{v}-\frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)v})y\right\} dy, \\
\beta_1(\theta) &= \frac{1}{\frac{1}{2}+\frac{h_1(\theta)^2}{2v}-\frac{\theta x}{v}-\frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)v}}, \\
\beta_2(\theta) &= \frac{1}{\frac{1}{2}+\frac{h_2(\theta)^2}{2v}-\frac{\theta x}{v}-\frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)v}}.
\end{aligned}
$$

Rewrite $I_1(\theta)$ and $I_2(\theta)$

$$\begin{aligned} I_1(\theta) &= \frac{\sqrt{v}\exp\{\psi(\theta)\}\Gamma(\frac{v-1}{2})\beta_1(\theta)^{\frac{v-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)\Gamma(\frac{v}{2})\beta^{v/2}}\int_{-\infty}^{\infty}\frac{1}{\Gamma(\frac{v-1}{2})\beta_1(\theta)^{\frac{v-1}{2}}}y^{(\frac{v-1}{2}-1)}\exp\{-\frac{y}{\beta_1(\theta)}\}dy \\ &= \frac{\sqrt{v}\exp\{\psi(\theta)\}\Gamma(\frac{v-1}{2})\beta_1(\theta)^{\frac{v-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)\Gamma(\frac{v}{2})\beta^{v/2}}, \end{aligned}$$

$$\begin{aligned} I_2(\theta) &= \frac{\sqrt{v}\exp\{\psi(\theta)\}\Gamma(\frac{v-1}{2})\beta_2(\theta)^{\frac{v-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)\Gamma(\frac{v}{2})\beta^{v/2}}\int_{-\infty}^{\infty}\frac{1}{\Gamma(\frac{v-1}{2})\beta_2(\theta)^{\frac{v-1}{2}}}y^{(\frac{v-1}{2}-1)}\exp\{-\frac{y}{\beta_2(\theta)}\}dy \\ &= \frac{\sqrt{v}\exp\{\psi(\theta)\}\Gamma(\frac{v-1}{2})\beta_2(\theta)^{\frac{v-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)\Gamma(\frac{v}{2})\beta^{v/2}}. \end{aligned}$$

Quantile estimation

Under the titling measure $\mathbb{P}^{\theta}$, the order of the portfolio return is

$$(R^*_{(1)}, \ldots, R^*_{(T)}), \tag{25}$$

where $R^*_{(1)}, \ldots, R^*_{(T)}$ are the order statistics of the sample $\{R^*_1, \ldots, R^*_T\}$. Therefore, an estimate of the quantile R^*_{1-p} for the portfolio return is

$$R^*_{(1-p)} \quad := \quad (1-s)R^*_{(j)} + sR^*_{(j+1)}. \tag{26}$$

where j is defined by

$$\frac{1}{N}\sum_{k=1}^{j} 1_{\{R_{(k)}<r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^{\theta}(R^*_{(k)})} < 1-p, \tag{27}$$

$$\frac{1}{N}\sum_{k=1}^{j+1} 1_{\{R_{(k)}<r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^{\theta}(R^*_{(k)})} > 1-p, \tag{28}$$

and s is defined by

$$s = (1 - p - \frac{1}{N} \sum_{k=1}^{j} 1_{\{R_{(k)} < r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^\theta(R^*_{(k)})}) \frac{d\mathbb{P}^\theta(R^*_{(j+1)})}{d\mathbb{P}(R^*_{(j+1)})}.$$

The asymptotic properties of the unbiased estimator for order statistics guarantee that as $n \to \infty$

$$\sqrt{n}(\hat{r}_p - r_p) \to N(0, \frac{p(1-p)}{f^2(r_p)}),$$

where $f = F'$ exists and is continuous at r_p. (cf. Hall, 1990, Johns, 1988, and Goffinet and Wallach, 1996). The decrease in variance of the classical estimator in place of the importance sampling estimator of the quantile will be the same as for p.

Consider parameters $\mu_1 = 0.01$, $\mu_2 = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.8$, $\rho_{12} = 0.3$, $\nu = 5$, sample size $T = 500$ and Monte Carlo replication $M = 10,000$. For simplicity, we set $w_i(t) = 1$ for all i and t. Table 1 reports the relative efficiency of quantile using naive Monte Carlo simulations and importance sampling for a multivariate normal distribution. The relative efficiency of the quantile estimation $\hat{r}_p(\theta)$ relative to the quantile estimation $\hat{r}_p$ is defined (cf. Hall (1991)) by

$$\text{eff}(\hat{r}_p(\theta), \hat{r}_p) = \frac{Var(\hat{r}_p)}{Var(\hat{r}_p(\theta))}, \tag{29}$$

where $Var(\hat{r}_p(\theta))$ is the variance of the quantile estimator $\hat{r}_p(\theta)$ with the parameter θ of importance sampling.

Table 1: The relative efficiency of the quantile estimation for a multivariate normal distribution in Monte Carlo simulation

r_p^N	0.5219	0.8013	1.1889	1.5091	2.1093	2.782
p	0.3000	0.2000	0.1000	0.0500	0.0100	0.001
$\hat{r}_p^N$	0.5150	0.7979	1.1727	1.4827	2.0739	2.465
$\widehat{se}_p^N$	5.19E-02	5.57E-02	6.68E-02	8.14E-02	0.1404	0.225
$\hat{r}_p^N(\theta_g^N)$	0.5227	0.8012	1.1884	1.5092	2.1090	2.782
$\widehat{se}_p^N(\theta_g^N)$	3.93E-02	3.95E-02	3.98E-02	3.96E-02	3.95E-02	3.97E-
$\mathrm{eff}(\widehat{r}_p^N(\theta_g^N), \widehat{r}_p^N)$	1.7439	1.9879	2.8208	4.2291	12.6258	32.28

Table 2: The relative efficiency of the quantile estimation for a multivariate t distribution in Monte Carlo simulation

r_p^T	0.5527	0.8698	1.3590	1.8350	3.0235	5.2450
p	0.3000	0.2000	0.1000	0.0500	0.0100	0.0010
$\hat{r}_p^T$	0.5449	0.8655	1.3392	1.7968	2.9527	4.1251
$\widehat{se}_p^T$	5.75E-02	5.46E-02	9.17E-02	0.1320	0.3367	0.8190
$\hat{r}_p^T(\theta_g^T)$	0.5533	0.8702	1.3594	1.8355	3.0228	5.2454
$\widehat{se}_p^T(\theta_g^T)$	5.26E-02	5.46E-02	6.16E-02	6.83E-02	9.27E-02	0.1434
$\mathrm{eff}(\widehat{r}_p^T(\theta_g^T), \hat{r}_p^T)$	1.1956	1.4591	2.2208	3.7381	13.1855	32.579

Table 3: Quadratic approximation function compared with the GHS method.
The parameters are $\nu = 5,\ k = 0,\ b = -1,\ \lambda = 0.5,\ T = 500$, and $M = 10000$.

	$x = 1$	$x = 2$	$x = 3$	$x = 5$
true P(A)	2.69E-01	1.47E-01	8.78E-02	3.80E-02
naive	2.69E-01	1.47E-01	8.77E-02	3.80E-02
variance	3.94E-04	2.52E-04	1.62E-04	7.30E-05
I.S. $\hat{p}$	2.69E-01	1.47E-01	8.78E-02	3.80E-02
variance	2.24E-04	9.70E-05	4.26E-05	9.85E-06
θ	4.56E-01	5.30E-01	5.83E-01	6.48E-01
GHS $\hat{p}$	2.69E-01	1.47E-01	8.78E-02	3.80E-02
variance	2.56E-04	1.04E-04	4.33E-05	9.92E-06
θ	2.16E-01	4.08E-01	5.00E-01	5.93E-01
relative efficiency	1.14	1.07	1.02	1.01

Table 4: Quadratic approximation function compared with the GHS method.
The parameters are
$\nu = 5,\ k = 0,\ b_1 = 0,\ b_2 = -1.183,\ \lambda_1 = 0.247,\ \lambda_2 = 0.147,\ T = 500,$ and $M = 10000$.

	$x = 1$	$x = 2$	$x = 3$	$x = 5$
true P(A)	3.16E-01	1.56E-01	8.06E-02	2.71E-02
naive $\hat{p}$	3.16E-01	1.56E-01	8.05E-02	2.70E-02
variance	4.22E-04	2.69E-04	1.48E-04	5.26E-05
I.S. $\hat{p}$	3.16E-01	1.56E-01	8.06E-02	2.70E-02
variance	2.06E-04	7.38E-05	2.48E-05	3.49E-06
θ	5.47E-01	8.94E-01	1.14E+00	1.37E+00
GHS $\hat{p}$	3.16E-01	1.56E-01	8.07E-02	2.70E-02
variance	2.49E-04	7.91E-05	2.53E-05	3.51E-06
θ	3.11E-01	6.48E-01	8.64E-01	1.12E+00
relative efficiency	1.21	1.07	1.02	1.00

Empirical study

We also analyze option returns from Ivy DB OptionMetrics, including the call option returns of “MICROSOFT CORP(MSFT)”, put option returns of “AT& T(T)”, and call option returns of “LUCENT TECHNOLOGIES INC(LU)”. The sample period is drawn from January 2, 2004, to January 21, 2005, and includes 266 observations. All excise dates are the same on January 22, 2005. We assume the returns follow a multivariate t distribution with the degree of freedom $\nu = 6$ and use the quadratic approximation method with standard sensitivities (delta, gamma). Hence the event $Q(R^T(t)) > 0$ can be computed in equation, given the estimate and standard error of the tail probability $p = 0.05$, and the bootstrap algorithm with importance resampling for replications $B = 200$, the VaR are 0.1528 and 3.87E$-$03.

Table 5: The statistics of “MSFT(call)”, “T(put)” and “LU(call)”

Company	Mean	Standard Deviation	Skewness	Kurtosis
MSFT(call)	-0.0264%	2.7943%	-1.3797	5.9770
T(put)	-0.0980%	2.9305%	1.8324	6.1437
LU(call)	0.1333%	7.0605%	0.7929	6.0522

Further work

- This paper considered the situation in which the change in risk factors has a multivariate t distribution. A possible shortcoming is that they require all X^i to share a parameter ν and thus have equally heavy tails. We can extend the model to allow **multiple degrees of freedom** and use a copula to do it (cf. Nelson, 1999 and Embrechts, 1999), but leave this for further studies.
- we can also change our model to the multivariate **jump diffusion model** or others to capture the heavy tails.
- Consider the **nonparametric bootstrap method** with importance resampling to evaluate VaR.