Efficient estimation using stock and option data under GARCH(1,1) model

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Agenda

- Introduction
- The model for stock and option data
- Main Theorem
- Simulation results
- Conclusion

Introduction

- Will using stock and option data to estimate the volatility have more accuracy?

- If using additional option data has better estimation, how many values will it raise?

- Using option data by option price formula or option price error, which one is better estimation?

  - using option price formula

    Jun Pan (2002) and Yacine Aït-Sahalia and Robert Kimmel (2007)

  - using option price error

    Bjørn Eraker (2004) and Michael Johannes, Nicholas Polson, and Jonathan Stroud (2008)

The model for stock data

- The stock data generating process under GARCH(1,1) model:

$$
\begin{aligned}
y_t &= \ln S_t - \ln S_{t-1} = h_t^{1/2} u_t,\ t = 1,\ldots,T, \\
h_t &= \omega + \alpha y_{t-1}^2 + \beta h_{t-1},\ t = 2,\ldots,T, \\
h_1 &= \omega,
\end{aligned}
$$

where $\{u_t\}_{t\in\mathbb{Z}}$ is $i.i.d.$, with $E(u_t) = 0$ and $E(u_t^2) = 1$, $E(u_t^4) < \infty$, $\omega > 0$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$, and assume that h_t is independent of $\{u_t, u_{t+1}, \ldots,\}$.

Quasi-maximum likelihood function for the stock data

- $\{y_t = \ln S_t - \ln S_{t-1},\ t = 0, 1, \ldots, T\}$ are observed data.

- Quasi-maximum likelihood function for the stock data is

$$L_T^X(y_0, \ldots, y_T; \theta) = L_T^X(\theta)$$

$$= \frac{-1}{2T} \sum_{t=1}^{T} \left(\ln\left(h_t(\theta)\right) + \frac{y_t^2}{h_t(\theta)} \right), \text{ where } \theta = (\omega,\ \alpha,\ \beta).$$

Option price formula

- Option price under GARCH(1,1) model, Jin-Chuan Duan(1995):

$$C_t^{BS,G}(T^*, K, S_t, h_t; \theta) = S_t \Phi(d_{1t}(\theta)) - e^{-r(T^*-t)} K \Phi(d_{2t}(\theta)), \qquad t = 1, \ldots, T,$$

where $T^*(>T)$ is maturity date,

K is excercise price,

$$d_{1t}(\theta) = \frac{\ln(S_t/K) + (r + \sigma^2(\theta)/2)(T^*-t)}{\sigma(\theta)\sqrt{T^*-t}},$$

$$d_{2t}(\theta) = d_{1t}(\theta) - \sigma(\theta)\sqrt{T^*-t},$$

and $\sigma^2(\theta) = E[h_t(\theta)] = \frac{\omega}{1-\alpha-\beta}.$

The model for option data

Case I: The option data = option price formula

$$C_t = C_t^{BS,G}, t = 1, \ldots, T.$$

Case II: The option data = option price formula + error term

$$C_t = C_t^{BS,G} + \eta z_t, t = 1, \ldots, T.$$

where $\eta > 0$ and $z_t \overset{i.i.d.}{\sim} N(0,1).$

Quasi-maximum likelihood function for stock and option data

Case I: the option data using option price formula

- Aït-sahalia and Robert (2007) provide a treatment of log-likelihood function for stock and option price.

Let $X_t = [S_t, h_t]'$ and $G_t = [S_t, C_t^{BS,G}]'$.

Then $G_{t+\Delta} = f(X_{t+\Delta};\theta) \Rightarrow X_{t+\Delta} = f^{-1}(G_{t+\Delta};\theta)$.

$$\Rightarrow P_G(\Delta, g \mid g_0;\theta) = det\left(\frac{\partial f(f^{-1}(g;\theta))}{\partial x}\right)^{-1} P_X(\Delta, f^{-1}(g;\theta) \mid f^{-1}(g_0;\theta);\theta)$$

$$= J_t(\Delta, g \mid g_0;\theta)^{-1} P_X(\Delta, f^{-1}(g;\theta) \mid f^{-1}(g_0;\theta);\theta),$$

$$\text{where } J_t(\Delta, g \mid g_0;\theta) = \begin{vmatrix} \frac{\partial S_t}{\partial S_t} & \frac{\partial S_t}{\partial h_t} \\ \frac{\partial C_t^{BS,G}}{\partial S_t} & \frac{\partial C_t^{BS,G}}{\partial h_t} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{\partial C_t^{BS,G}}{\partial S_t} & \frac{\partial C_t^{BS,G}}{\partial h_t} \end{vmatrix} = \left| \frac{\partial C_t^{BS,G}}{\partial h_t}(\Delta, g \mid g_0;\theta) \right|,$$

$g = G_{t+\Delta}$, and $g_0 = G_t$.

$$\Rightarrow l_G(\Delta, g \mid g_0;\theta) \equiv \ln P_G(\Delta, g \mid g_0;\theta) = -\ln J_t(\Delta, g \mid g_0;\theta) + l_X(\Delta, f^{-1}(g;\theta) \mid f^{-1}(g_0;\theta);\theta).$$

$$\Rightarrow L_T^G(\theta) = \frac{1}{T}\sum_{t=1}^{T} l_G(G_{t+1} \mid G_t;\theta) = \frac{1}{T}\sum_{t=1}^{T}\left[-\ln J_t(G_{t+1} \mid G_t;\theta) + l_X(X_{t+1} \mid X_t;\theta)\right].$$

Quasi-maximum likelihood function for stock and option data

Case I: the option data using option price formula

$$J_t(\theta)=\left|\frac{\partial C^{BS,G}(t,T^*,K,S_t,h_t;\theta)}{\partial h_t}\right|$$

$$=\frac{e^{-r(T^*-t)}K\sqrt{T^*-t}\,\phi(d_{2t}(\theta))}{2\sigma(\theta)}$$

$$=\frac{e^{-r(T^*-t)}K\sqrt{T^*-t}}{2\sigma(\theta)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_{2t}^2(\theta)}{2}}$$

$$\Rightarrow -\ln J_t(\theta)=-\ln\left(\frac{e^{-r(T^*-t)}\sqrt{T^*-t}K}{2\sqrt{2\pi}}\right)+\frac{d_{2t}^2(\theta)+\ln\left(\sigma^2(\theta)\right)}{2}$$

$$\Rightarrow L_T^G(\theta)=\frac{1}{T}\sum_{t=1}^{T}l_G(G_{t+1}\,|\,G_t;\theta)=\frac{1}{T}\sum_{t=1}^{T}[-\ln J_t(G_{t+1}\,|\,G_t;\theta)+l_X(X_{t+1}\,|\,X_t;\theta)]$$

$$=\frac{1}{2T}\sum_{t=1}^{T}\left[d_{2t}^2(\theta)+\ln\left(\sigma^2(\theta)\right)-\left(\ln\left(h_t(\theta)\right)+\frac{y_t^2}{h_t(\theta)}\right)\right]$$

Quasi-maximum likelihood function for stock and option data

Case II: the option data using option price + error term

Suppose that $\operatorname{cor}(u_t, z_t) = \rho$, where $|\rho| < 1$ and $t = 1, \ldots, T$.

Then $\begin{pmatrix} u_t \\ z_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$.

$\Rightarrow$ The conditional density of G_t given G_{t-1} is

$$\frac{1}{2\pi\sqrt{1-\rho^2}\sqrt{h_t(\theta)}\eta} exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{y_t^2}{h_t(\theta)} - 2\rho \frac{y_t}{\sqrt{h_t(\theta)}} \frac{C_t - C_t^{BS,G}(\theta)}{\eta} + \frac{\left(C_i - C_t^{BS,G}(\theta)\right)^2}{\eta^2} \right) \right].$$

$$\Rightarrow L_T^G(\theta) = \frac{-1}{2T} \sum_{t=1}^{T} \left[\ln(h_t(\theta)) + \frac{1}{2(1-\rho^2)} \left(\frac{y_t^2}{h_t(\theta)} - 2\rho \frac{y_t}{\sqrt{h_t(\theta)}} \frac{C_t - C_t^{BS,G}(\theta)}{\eta} + \frac{\left(C_i - C_t^{BS,G}(\theta)\right)^2}{\eta^2} \right) \right]$$

Quasi-maximum likelihood estimators of the parameters

- **Only stock**

$$\hat{\theta}_T^X = \max_{\theta} L_T^X(\theta), \text{ where } L_T^X(\theta) = \frac{-1}{2T}\sum_{t=1}^{T}\left(\ln\left(h_t(\theta)\right) + \frac{y_t^2}{h_t(\theta)}\right).$$

- **Stock and option price**

$$\hat{\theta}_T^{G1} = \max_{\theta} L_T^{G1}(\theta), \text{ where } L_T^{G1}(\theta) = \frac{-1}{2T}\sum_{t=1}^{T}\left[-\left(d_{2t}^2(\theta) + \ln(\sigma^2(\theta))\right) + \left(\ln\left(h_t(\theta)\right) + \frac{y_t^2}{h_t(\theta)}\right)\right].$$

- **Stock and option price + error**

$$\hat{\theta}_T^{G2} = \max_{\theta} L_T^{G2}(\theta),$$

$$\text{where } L_T^{G2}(\theta) = \frac{-1}{2T}\sum_{t=1}^{T}\left[\ln(h_t(\theta)) + \frac{1}{2(1-\rho^2)}\left(\frac{y_t^2}{h_t(\theta)} - 2\rho\frac{y_t}{\sqrt{h_t(\theta)}}\frac{C_t - C_t^{BS,G}(\theta)}{\eta} + \frac{\left(C_i - C_t^{BS,G}(\theta)\right)^2}{\eta^2}\right)\right].$$

Main Theorem

- **Only stock, *Robin L. Lumsdaine* (1996)**

(i) consistent: $\hat{\theta}_T^X \xrightarrow{p} \theta_0$.

(ii) asymptotically normal: $B_{X0}^{-1/2} A_{X0} T^{1/2} (\hat{\theta}_T^X - \theta_0) \overset{A}{\sim} N(0, I)$,

$$\text{where } A_{X0} = -E\left(\frac{\partial^2 L_T^X(\theta_0)}{\partial\theta\partial\theta'} \right) \text{ and } B_{X0} = E\left(T \frac{\partial L_T^X(\theta_0)}{\partial\theta} \frac{\partial L_T^X(\theta_0)}{\partial\theta'} \right).$$

- **Stock and option price**

(i) bias : $\hat{\theta}_T^{G1} - \theta_1 \xrightarrow{p} 0$, where $\theta_1 \equiv \theta_0 + A_{G10}^{-1} E[\frac{\partial L_T^{G1}(\theta_0)}{\partial\theta}]$.

(ii) asymptotically normal: $B_{G10}^{-1/2} A_{G10} T^{1/2} (\hat{\theta}_T^{G1} - \theta_1) \overset{A}{\sim} N(0, I)$,

$$\text{where } A_{G10} = -E\left(\frac{\partial^2 L_T^{G1}(\theta_0)}{\partial\theta\partial\theta'} \right) \text{ and } B_{G10} = Var\left(T^{1/2} \frac{\partial L_T^{G1}(\theta_0)}{\partial\theta} \right).$$

- **Stock and option price + error**

(i) consistent: $\hat{\theta}_T^{G2} \xrightarrow{p} \theta_0$.

(ii) asymptotically normal: $B_{G20}^{-1/2} A_{G20} T^{1/2} (\hat{\theta}_T^{G2} - \theta_0) \overset{A}{\sim} N(0, I)$,

$$\text{where } A_{G20} = -E\left(\frac{\partial^2 L_T^{G2}(\theta_0)}{\partial\theta\partial\theta'} \right) \text{ and } B_{G20} = E\left(T \frac{\partial L_T^{G2}(\theta_0)}{\partial\theta} \frac{\partial L_T^{G2}(\theta_0)}{\partial\theta'} \right).$$

Compare with fisher information

- Suppose that the distribution of the errors (u_t) is truly normal.
- **Only stock**

$$A_{X0} = B_{X0} = \frac{1}{2T}\sum_{t=1}^{T} E[\frac{1}{h_t^2(\theta_0)}\frac{\partial h_t(\theta_0)}{\partial \theta}\frac{\partial h_t(\theta_0)}{\partial \theta'}].$$

- **Stock and option price**

$$A_{G10} = A_{X0} + I_1(\theta_0),\ \text{where}\ I_1(\theta_0) = \frac{a(\theta_0)}{\omega_0^2}\begin{pmatrix} 1 & \sigma^2(\theta_0) & \sigma^2(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \end{pmatrix} + \frac{b(\theta_0)}{\omega_0^2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2\sigma^2(\theta_0) & 2\sigma^2(\theta_0) \\ 1 & 2\sigma^2(\theta_0) & 2\sigma^2(\theta_0) \end{pmatrix},$$

$$a(\theta_0) = \frac{1}{2T}\sum_{t=1}^{T}\left[1 - \frac{2t}{T^* - t} - \frac{2\left[r(T^* - t) + \ln(S_0/K)\right]^2}{\sigma^2(\theta_0)\left(T^* - t\right)}\right],\ \text{and}\ b(\theta_0) = \frac{1}{2T}\sum_{t=1}^{T}\left[\frac{t}{T^* - t} + \frac{\left[r(T^* - t) + \ln(S_0/K)\right]^2}{\sigma^2(\theta_0)(T^* - t)} - \frac{\sigma^2(\theta_0)}{4}(T^* - t) - 1\right].$$

$$B_{G10} = A_{X0} + I_2(\theta_0),\ \text{where}\ I_2(\theta_0) = \frac{c(\theta_0)}{\omega_0^2}\begin{pmatrix} 1 & \sigma^2(\theta_0) & \sigma^2(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \end{pmatrix}\ \text{and}\ c(\theta_0) = \frac{1}{2T}\sum_{t=1}^{T}\left[2\frac{t}{T^* - t}\frac{\left(r(T^* - t) + \ln(S_0/K)\right)^2}{\sigma^2(\theta_0)\left(T^* - t\right)} + \left(\frac{t}{T^* - t}\right)^2\right] > 0.$$

- **Stock and option price + error**

$$A_{G20} = B_{G20} = A_{X0} + I_3(\theta_0),\text{where}\ I_3(\theta_0) = \frac{\rho^2}{2(1-\rho^2)}A_{X0} + \frac{1}{(1-\rho^2)\eta^2}\frac{\sigma^2(\theta_0)m(\theta_0)}{\omega_0^2}\begin{pmatrix} 1 & \sigma^2(\theta_0) & \sigma^2(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \\ \sigma^2(\theta_0) & \sigma^4(\theta_0) & \sigma^4(\theta_0) \end{pmatrix}$$

$$\text{and}\ m(\theta_0) = \frac{1}{2T}\sum_{t=1}^{T}\frac{K^2(T^* - t)}{4\pi}\sqrt{\frac{T^* - t}{T^* + t}}\exp\left[-\frac{\left((r - \frac{\sigma^2(\theta_0)}{2})T^* + \ln(S_0/K)\right)^2}{\sigma^2(\theta_0)(T^* + t)} - 2r(T^* - t)\right] > 0.$$

Consistent estimate of A_{X0}

$$A_{X0} = \frac{1}{2T}\sum_{t=1}^{T} E\left[\frac{1}{h_t^2(\theta_0)}\frac{\partial h_t(\theta_0)}{\partial \theta}\frac{\partial h_t(\theta_0)}{\partial \theta'}\right].$$

Further, consistent estimate of A_{X0} is given by

$$\hat{A}_T^X = \frac{1}{2T}\sum_{t=1}^{T} \frac{1}{h_t^2(\theta_0)}\frac{\partial h_t(\theta_0)}{\partial \theta}\frac{\partial h_t(\theta_0)}{\partial \theta'}.$$

Compare with variance

- **Only stock**

$$V_X(\theta_0) = A_{X0}^{-1} B_{X0} A_{X0}^{-1} = A_{X0}^{-1}$$

- **Stock and option price**

$$V_{G1}(\theta_0) = A_{G10}^{-1} B_{G10} A_{G10}^{-1}$$

$$A_{G10} B_{G10}^{-1} A_{G10} - A_{X0} = \left(A_{X0} + I_1(\theta_0)\right)\left(A_{X0} + I_2(\theta_0)\right)^{-1}\left(A_{X0} + I_1(\theta_0)\right) - A_{X0}$$

- **Stock and option price + error**

$$V_{G2}(\theta_0) = A_{G20}^{-1} B_{G20} A_{G20}^{-1} = A_{G20}^{-1}$$

$$A_{G20} - A_{X0} = I_3(\theta_0)$$

Example 1

- If ω is only unknown, then
- Only stock

(i) consistent: $\hat{\omega}_T^X \xrightarrow{p} \omega_0$.

(ii) asymptotically normal: $V_X^{-1/2}(\omega_0)T^{1/2}(\hat{\omega}_T^X - \omega_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_X^{-1}(\omega_0) = A_{X0}(\omega_0) = \frac{1}{2T}\sum_{t=1}^{T} E\left[\frac{1}{h_t^2(\omega_0)}\left(\frac{\partial h_t(\omega_0)}{\partial\omega}\right)^2\right].$$

- Stock and option price

(i) bias: $\hat{\omega}_T^{G1} - \omega_1 \xrightarrow{p} 0$, where $\omega_1 = \omega_0 + A_{G10}^{-1}(\omega_0)E[\frac{\partial L_T^{G1}(\omega_0)}{\partial\omega}]$.

(ii) asymptotically normal: $V_{G1}^{-1/2}(\omega_0)T^{1/2}(\hat{\omega}_T^{G1} - \omega_1) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G1}^{-1}(\omega_0) = A_{G10}(\omega_0)B_{G10}^{-1}(\omega_0)A_{G10}(\omega_0) = \left(A_{X0}(\omega_0) + \frac{a(\omega_0)}{\omega_0^2}\right)^2\left(A_{X0}(\omega_0) + \frac{c(\omega_0)}{\omega_0^2}\right)^{-1}$$

$$\text{and } A_{G10}(\omega_0)B_{G10}^{-1}(\omega_0)A_{G10}(\omega_0) - A_{X0}(\omega_0) = \left(\frac{a^2(\omega_0)}{\omega_0^4} + \frac{2a(\omega_0) - c(\omega_0)}{\omega_0^2}A_{X0}(\omega_0)\right)\left(A_{X0}(\omega_0) + \frac{c(\omega_0)}{\omega_0^2}\right)^{-1}.$$

- Stock and option price + error

(i) consistent: $\hat{\omega}_T^{G2} \xrightarrow{p} \omega_0$.

(ii) asymptotically normal: $V_{G2}^{-1/2}(\omega_0)T^{1/2}(\hat{\omega}_T^{G2} - \omega_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G2}^{-1}(\omega_0) = A_{G20}(\omega_0) = A_{X0}(\omega_0) + I_3(\omega_0) \text{ and } I_3(\omega_0) = \frac{\rho^2}{2(1-\rho^2)}A_{X0}(\omega_0) + \frac{1}{(1-\rho^2)\eta^2}\frac{\sigma^2(\omega_0)m(\omega_0)}{\omega_0^2} > 0.$$

Example 2

- If α is only unknown, then
- Only stock

(i) consistent: $\hat{\alpha}_T^X \xrightarrow{p} \alpha_0$.

(ii) asymptotically normal: $V_X^{-1/2}(\alpha_0)T^{1/2}(\hat{\alpha}_T^X - \alpha_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_X^{-1}(\alpha_0) = A_{X0}(\alpha_0) = \frac{1}{2T}\sum_{t=1}^{T} E\left[\frac{1}{h_t^2(\alpha_0)}\left(\frac{\partial h_t(\alpha_0)}{\partial \alpha}\right)^2\right].$$

- Stock and option price

(i) bias: $\hat{\alpha}_T^{G1} - \alpha_1 \xrightarrow{p} 0$, where $\alpha_1 = \alpha_0 + A_{G10}^{-1}(\alpha_0)E\left[\frac{\partial L_T^{G1}(\alpha_0)}{\partial \alpha}\right]$.

(ii) asymptotically normal: $V_{G1}^{-1/2}(\alpha_0)T^{1/2}(\hat{\alpha}_T^{G1} - \alpha_1) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G1}^{-1}(\alpha_0) = A_{G10}(\alpha_0)B_{G10}^{-1}(\alpha_0)A_{G10}(\alpha_0) = \left(A_{X0}(\alpha_0) + \frac{\sigma^4(\alpha_0)a(\alpha_0) + 2\sigma^2(\alpha_0)b(\alpha_0)}{\omega_0^2}\right)^2\left(A_{X0}(\alpha_0) + \frac{\sigma^4(\alpha_0)c(\alpha_0)}{\omega_0^2}\right)^{-1}$$

$$\text{and } V_{G1}^{-1}(\alpha_0) - A_{X0}(\alpha_0) = \left[\left(\frac{\sigma^4(\alpha_0)a(\alpha_0) + 2\sigma^2(\alpha_0)b(\alpha_0)}{\omega_0^2}\right)^2 + \frac{\sigma^4(\alpha_0)\left(2a(\alpha_0) - c(\alpha_0)\right) + 4\sigma^2(\alpha_0)b(\alpha_0)}{\omega_0^2}A_{X0}(\alpha_0)\right]\left(A_{X0}(\alpha_0) + \frac{\sigma^4(\alpha_0)c(\alpha_0)}{\omega_0^2}\right)^{-1}.$$

- Stock and option price + error

(i) consistent: $\hat{\alpha}_T^{G2} \xrightarrow{p} \alpha_0$.

(ii) asymptotically normal: $V_{G2}^{-1/2}(\alpha_0)T^{1/2}(\hat{\alpha}_T^{G2} - \alpha_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G2}^{-1}(\alpha_0) = A_{G20}(\alpha_0) = A_{X0}(\alpha_0) + I_3(\alpha_0) \text{ and } I_3(\alpha_0) = \frac{\rho^2}{2(1-\rho^2)}A_{X0}(\alpha_0) + \frac{1}{(1-\rho^2)\eta^2}\frac{\sigma^6(\alpha_0)m(\alpha_0)}{\omega_0^2} > 0.$$

Example 3

- If β is only unknown, then
- Only stock

(i) consistent: $\hat{\beta}_T^X \xrightarrow{p} \beta_0$.

(ii) asymptotically normal: $V_X^{-1/2}(\beta_0)T^{1/2}(\hat{\beta}_T^X - \beta_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_X^{-1}(\beta_0) = A_{X0}(\beta_0) = \frac{1}{2T}\sum_{t=1}^{T} E[\frac{1}{h_t^2(\beta_0)}\left(\frac{\partial h_t(\beta_0)}{\partial \beta}\right)^2].$$

- Stock and option price

(i) bias: $\hat{\beta}_T^{G1} - \beta_1 \xrightarrow{p} 0$, where $\beta_1 = \beta_0 + A_{G10}^{-1}(\beta_0)E[\frac{\partial L_T^{G1}(\beta_0)}{\partial \beta}]$.

(ii) asymptotically normal: $V_{G1}^{-1/2}(\beta_0)\ T^{1/2}(\hat{\beta}_T^{G1} - \beta_1) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G1}^{-1}(\beta_0) = A_{G10}(\beta_0)B_{G10}^{-1}(\beta_0)A_{G10}(\beta_0) = \left(A_{X0}(\beta_0) + \frac{\sigma^4(\beta_0)a(\beta_0) + 2\sigma^2(\beta_0)b(\beta_0)}{\omega_0^2}\right)^2 \left(A_{X0}(\beta_0) + \frac{\sigma^4(\beta_0)c(\beta_0)}{\omega_0^2}\right)^{-1}$$

$$\text{and } V_{G1}^{-1}(\beta_0) - A_{X0}(\beta_0) = \left[\left(\frac{\sigma^4(\beta_0)a(\beta_0) + 2\sigma^2(\beta_0)b(\beta_0)}{\omega_0^2}\right)^2 + \frac{\sigma^4(\beta_0)\left(2a(\beta_0) - c(\beta_0)\right) + 4\sigma^2(\beta_0)b(\beta_0)}{\omega_0^2}A_{X0}(\beta_0)\right]\left(A_{X0}(\beta_0) + \frac{\sigma^4(\beta_0)c(\beta_0)}{\omega_0^2}\right)^{-1}.$$

- Stock and option price + error

(i) consistent: $\hat{\beta}_T^{G2} \xrightarrow{p} \beta_0$.

(ii) asymptotically normal: $V_{G2}^{-1/2}(\beta_0)T^{1/2}(\hat{\beta}_T^{G2} - \beta_0) \overset{A}{\sim} N(0,1)$,

$$\text{where } V_{G2}^{-1}(\beta_0) = A_{G20}(\beta_0) = A_{X0}(\beta_0) + I_3(\beta_0) \text{ and } I_3(\beta_0) = \frac{\rho^2}{2(1-\rho^2)}A_{X0}(\beta_0) + \frac{1}{(1-\rho^2)\eta^2}\frac{\sigma^6(\beta_0)m(\beta_0)}{\omega_0^2} > 0.$$

Simulation Results

Table 1-1

Mean values and standard deviations of the parameter estimates (at the money)

$N = 1{,}000$, $T = 90$, $r = 0.03$, $K = 100$, $S_0 = 100$, $(\omega_0, \alpha_0, \beta_0) = (0.008, 0.122, 0.854)$

Panel A : ω is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\omega - \omega_0\rvert$	mean	0.002408	0.08486	0.00009
	Std	0.00232	0.02359	0.000374

Panel B : α is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\alpha - \alpha_0\rvert$	mean	0.020916	0.023265	0.000525
	Std	0.016663	0.00374	0.002217

Panel C : β is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\beta - \beta_0\rvert$	mean	0.018205	0.023159	0.000525
	Std	0.01414	0.0029	0.002297

Simulation Results

Table 1-2

Mean values and standard deviations of the parameter estimates (at the money)

N = 1,000, T = 90, r = 0.03, K = 100, S0 = 100, (ω_0, α_0, β_0) = (0.008, 0.122, 0.854)

Panel A : ω is only unknown						
		Stock and option price + error				
		ρ = 0				
		η = 1	η = 0.5	η = 0.1	η = 0.01	η = 0.001
$\|\omega - \omega_0\|$	mean	0.00009	0.000025	0.000001	0.00E+00	0.00E+00
	Std	0.000374	0.000206	0.000032	0.00E+00	0.00E+00
Panel B : α is only unknown						
		Stock and option price + error				
		ρ = 0				
		η = 1	η = 0.5	η = 0.1	η = 0.01	η = 0.001
$\|\alpha - \alpha_0\|$	mean	0.000644	0.000217	0.000012	0.00E+00	0.00E+00
	Std	0.002754	0.001106	0.000118	0.00E+00	0.00E+00
Panel C : β is only unknown						
		Stock and option price + error				
		ρ = 0				
		η = 1	η = 0.5	η = 0.1	η = 0.01	η = 0.001
$\|\beta - \beta_0\|$	mean	0.000525	0.000182	0.000003	0.00E+00	0.00E+00
	Std	0.002297	0.000939	0.000055	0.00E+00	0.00E+00

Simulation Results

Table 1-3

Mean values and standard deviations of the parameter estimates (at the money)

N = 1,000, T = 90, r = 0.03, K = 100, S0 = 100, (ω0, α0, β0) = (0.008, 0.122, 0.854)

Panel A : ω is only unknown

		Stock and option price + error					
		η = 1					
		ρ = 0	ρ = 0.1	ρ = 0.25	ρ = 0.5	ρ = 0.75	ρ = 0.9
$\|\omega - \omega_0\|$	mean	0.000175	0.000151	0.000128	0.000224	0.001335	0.005447
	Std	0.001093	0.00078	0.000669	0.001465	0.008126	0.020461

Panel B : α is only unknown

		Stock and option price + error					
		η = 1					
		ρ = 0	ρ = 0.1	ρ = 0.25	ρ = 0.5	ρ = 0.75	ρ = 0.9
$\|\alpha - \alpha_0\|$	mean	0.000548	0.000552	0.000462	0.000709	0.000737	0.000977
	Std	0.002301	0.002273	0.001883	0.002891	0.003017	0.003526

Panel C : β is only unknown

		Stock and option price + error					
		η = 1					
		ρ = 0	ρ = 0.1	ρ = 0.25	ρ = 0.5	ρ = 0.75	ρ = 0.9
$\|\beta - \beta_0\|$	mean	0.000583	0.000475	0.000465	0.00067	0.000877	0.001029
	Std	0.002175	0.002005	0.002088	0.002699	0.003413	0.003817

Simulation Results

Table 2

Mean values and standard deviations of the parameter estimates (out of the money)

$N = 1{,}000$, $T = 90$, $r = 0.03$, $\mathbf{K = 110}$, $S_0 = 100$, $(\omega_0, \alpha_0, \beta_0) = (0.008, 0.122, 0.854)$

Panel A : ω is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\omega - \omega_0\rvert$	mean	0.002446	0.084775	0.000106
	Std	0.002678	0.023717	0.000594

Panel B : α is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\alpha - \alpha_0\rvert$	mean	0.021737	0.023212	0.000502
	Std	0.017009	0.003347	0.002228

Panel C : β is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\beta - \beta_0\rvert$	mean	0.017616	0.023212	0.000409
	Std	0.013725	0.003347	0.001842

Simulation Results

Table 3

Mean values and standard deviations of the parameter estimates (in the money)

N = 1,000, T = 90, r = 0.03, **K = 90**, S_0 = 100, (ω_0, α_0, β_0) = (0.008, 0.122, 0.854)

Panel A : ω is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\omega - \omega_0\rvert$	mean	0.002438	0.083755	0.000162
	Std	0.002563	0.025169	0.000888

Panel B : α is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\alpha - \alpha_0\rvert$	mean	0.021847	0.023159	0.000487
	Std	0.017288	0.0029	0.001801

Panel C : β is only unknown

		Only stock	Stock and option price	Stock and option price + error
				$\eta = 1, \rho = 0$
$\lvert\beta - \beta_0\rvert$	mean	0.018144	0.023265	0.000424
	Std	0.014403	0.00374	0.001416

Simulation Results

Table 4

Fisher information and standard deviations of the parameter estimates

$N = 1{,}000$, $T = 90$, $r = 0.03$, $K = 100$, $S_0 = 100$, $(\omega_0, \alpha_0, \beta_0) = (0.008, 0.122, 0.854)$

Panel A : ω is only unknown

	Only stock	Stock and option price	Stock and option price + error $\eta = 1, \rho = 0$
Fisher information	1578.1	-57201	113240
1 / variance	1578.1	2959.9	113240
	0	1381.8	111661.9
std	0.002653454	0.001937493	0.000313241

Panel B : α is only unknown

	Only stock	Stock and option price	Stock and option price + error $\eta = 1, \rho = 0$
Fisher information	13.9417	-9276.4	12421
1 / variance	13.9417	701.4656	12421
	0	687.5239	12407.0583
std	0.02823065	0.003979931	0.000945803

Panel C : β is only unknown

	Only stock	Stock and option price	Stock and option price + error $\eta = 1, \rho = 0$
Fisher information	19.3501	-9270.9	12426
1 / variance	19.3501	700.617	12426
	0	681.2669	12406.6499
std	0.023962777	0.003982341	0.000945612

Conclusion

- We show that the asymptotically theory of stock and option data under GARCH(1,1) model.

- Using stock and option price error is better estimation than both using only stock and using stock and option price.

- Using stock and option price has asymptotically bias and is not better estimation for ω than using only stock.

- Compared with fisher information matrix and variance, using additional option data has more improvement.

Thank you for your attentions