An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process

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Outline

Introduction

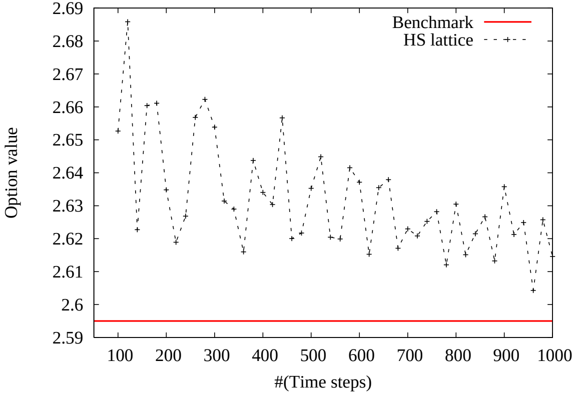
- A derivative product is financial instrument whose payoff is based on other underlying assets such as stocks.
- Pricing it is equivalent to computing its expected payoff under a suitable probability measure.
- Most derivatives have no analytical formulas.
- So they must be priced by numerical methods like the lattice model.

Introduction

- However, the nonlinearity error may cause the pricing results to converge slowly.
- It may even cause the pricing results to oscillate significantly.[1]
- The goals are numerical accuracy and speed.

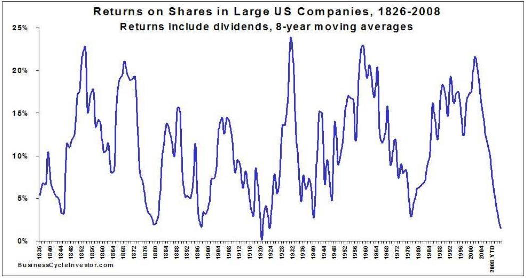
[1] Figlewski and Gao (1999).

Oscillations



Models

- Lognormal diffusion process has been widely used to model the underlying asset's price dynamics.[2]
- Unfortunately, the lognormal diffusion process is incapable of capturing empirical stock price behaviors.[3]
- Many alternative processes, such as GARCH process, jump-diffusion process, have been proposed.



[2] Black and Scholes (1973).

Related Work

- Amin (1993)
  - He approximates the jump-diffusion process by a multinomial lattice.
  - Huge numbers of branches at each node make the lattice inefficient.
- Hilliard and Schwartz (2005)
  - They match the first local moments of the lognormal jumps.
  - Their lattice lacks the flexibility to suit derivatives' specifications.

Main Results

- This talk proposes an efficient lattice model for the jump-diffusion process.
- The time complexity of our lattice is $O(n^{2.5})$.
- Our lattice is adjusted to suit the derivatives' specification so that the price oscillation problem can be significantly suppressed.

Jump-Diffusion Process

- Define S_t as the stock price at time t.
- Merton's jump-diffusion model assumes that the stock price process can be expressed as

$$S_t = S_0 e^{(r-\lambda\bar{k}-0.5\sigma^2)t+\sigma z(t) U(n(t)))}. \tag{1}$$

  - $z(t)$ denotes a standard Brownian motion.
  - r denotes the risk-free rate.
  - σ denotes the volatility of the diffusion component of the stock price process.
  - $U(n(t)) = \prod_{i=0}^{n(t)} (1+k_i)$ and $k_0 = 0$.

Jump-Diffusion Process

- Jump events are governed by the Poisson process $n(t)$ with jump intensity λ.
- The random jump magnitude k_i $(i > 0)$ follows the equation:

$$\ln(1 + k_i) \sim N(\gamma, \delta),$$

where $E(k_i) \equiv \bar{k} = e^{\gamma + 0.5\delta^2} - 1$.

Jump-Diffusion Process

- Hilliard and Schwartz decompose S_0-log-price of S_t into the diffusion component and the jump component by rewriting Eq. (1) as follows:

$$V_t \equiv \ln\left(\frac{S_t}{S_0}\right) = X_t + Y_t,$$

  - The diffusion component

$$X_t \equiv \left(r - \lambda\bar{k} - 0.5\sigma^2\right) t + \sigma z(t)$$

    is a Brownian motion.
  - The jump component

$$Y_t \equiv \sum_{i=0}^{n(t)} \ln\left(1 + k_i\right)$$

    is normal under Poisson compounding.

Financial Background

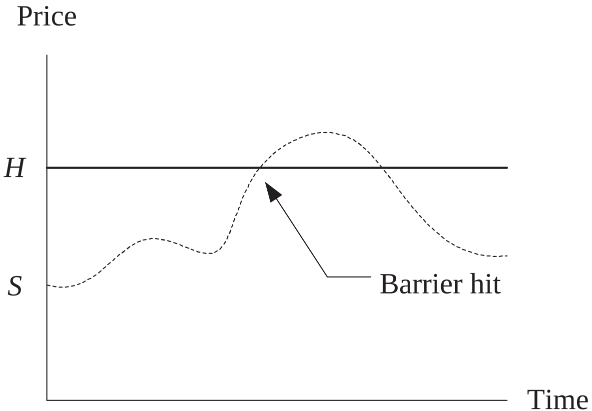
- An option is a financial instrument.
- It represents a *right* to buy the stock for a price (the exercise price X) at the maturity date T.
- Options are essential to speculation and the management of financial risk.[4]
- The payoff of a European-style vanilla option at the maturity date T is $\max(S_T - X, 0)$.
- The payoff of an American-style vanilla option at time t $(0 \leq t \leq T)$ is $S_t - X$.

[4] Hull (2002).

Financial Background

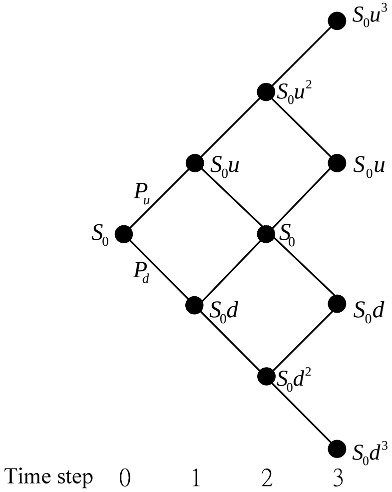
- A barrier option is similar to a vanilla option.
- But the payoff of a barrier option depends on whether the underlying stock's price path ever touches the barrier(s).
  - Such options are very popular in the financial market.

Barrier Options



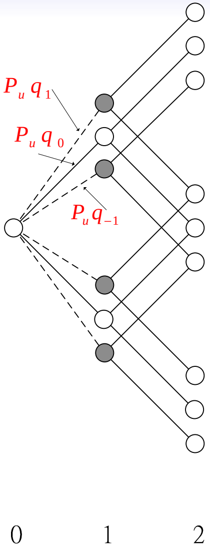
CRR Lattice for the Diffusion Part

- The size of one time step is $\Delta t = T/n$.
- u, d, P_u, P_d:
  - Match the mean and variance of the stock return.
  - $ud = 1$.
  - $P_u + P_d = 1$.



Hilliard and Schwartz's Lattice

- Diffusion part (X_t)
  - Match mean and variance of $X_{\Delta t}$.
  - Obtain P_u and P_d.
- Jump part (Y_t)
  - Match the first $2m$ local moments of $Y_{\Delta t}$.
  - Obtain q_j $(j = 0, \pm 1, \pm 2, \ldots, \pm m)$.



Hilliard and Schwartz's Lattice

- The stock price of the node at time step i is $S_0^{V_{i\Delta t}}$.
- Motivated by the decomposition in Eq. (11), $V_{(i+1)\Delta t}$ can be represented by

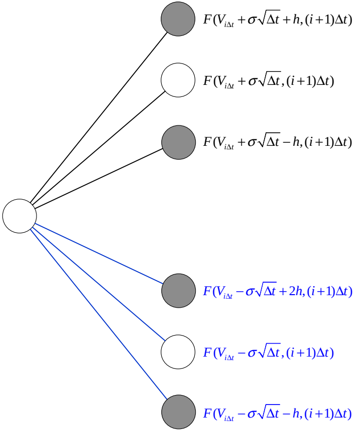
$$V_{(i+1)\Delta t} = V_{i\Delta t} + c\sigma\sqrt{\Delta t} + jh,\ j = 0, \pm 1, \pm 2, \ldots, \pm m.$$

  - $c \in \{-1, 1\}$ denotes the upward or the downward movement of the stock price driven by the diffusion component.
  - j denotes the number of positions above or below the node $V_{i\Delta t} + c\sigma\sqrt{\Delta t}$.
  - The magnitude of the basic jump unit is set to $h = \sqrt{\gamma^2 + \delta^2}$.
- The node count of the lattice is $O(n^3)$.

Hilliard and Schwartz's Lattice

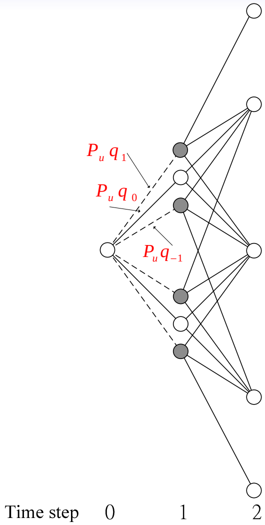
- Define $F(V_{i\Delta t}, i)$ as the option value (stock price is $S_0 e^{V_{i\Delta t}}$).
- European-style options

$$F(V_{i\Delta t}, i) =$$
$$e^{-r\Delta t} \sum_{j=-m}^{m} F\left(V_{i\Delta t} + \sigma\sqrt{\Delta t} + jh, (i+1)\,\Delta t\right) P_u q_j +$$
$$e^{-r\Delta t} \sum_{j=-m}^{m} F\left(V_{i\Delta t} - \sigma\sqrt{\Delta t} + jh, (i+1)\,\Delta t\right) P_d q_j .$$



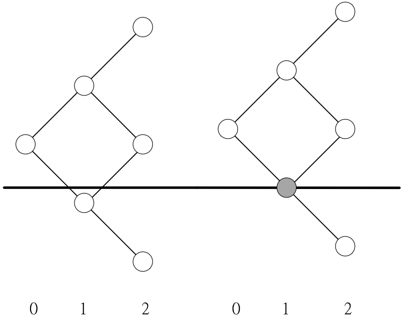
Our Lattice

- Diffusion part (X_t)
  - Follow a CRR structure.
  - Follow a trinomial structure.
- Jump part (Y_t)
  - Match the first $2m$ local moments of $Y_{\Delta t}$.
  - Obtain q_j $(j = 0, \pm 1, \pm 2, \ldots, \pm m)$.

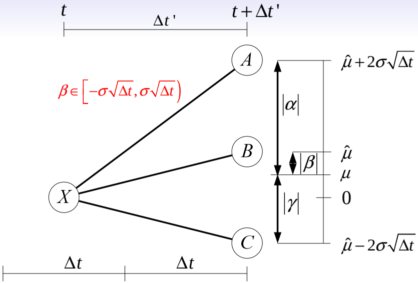


Price Oscillation Problem

- Price oscillation problem is mainly due to the nonlinearity error.
  - Introduced by the nonlinearity of the option value function.
- The solution of the nonlinearity error:
  - Making price level of the lattice coincide with the location where the option value function is highly nonlinear.



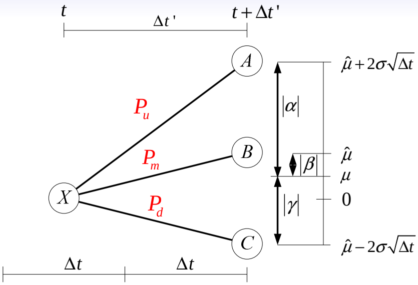
Trinomial Structure



The branching probabilities for the node X

$$
\begin{aligned}
\beta &\equiv \hat{\mu} - \mu, \\
\alpha &\equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}, \\
\gamma &\equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}, \\
\hat{\mu} &\equiv \ln\left(s(B)/s(X)\right).
\end{aligned}
$$

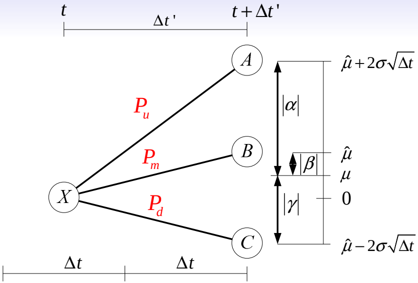
Trinomial Structure



The branching probabilities for the node X

$$P_u\alpha + P_m\beta + P_d\gamma = 0,$$
$$P_u(\alpha)^2 + P_m(\beta)^2 + P_d(\gamma)^2 = \text{Var},$$
$$P_u + P_m + P_d = 1.$$

Trinomial Structure

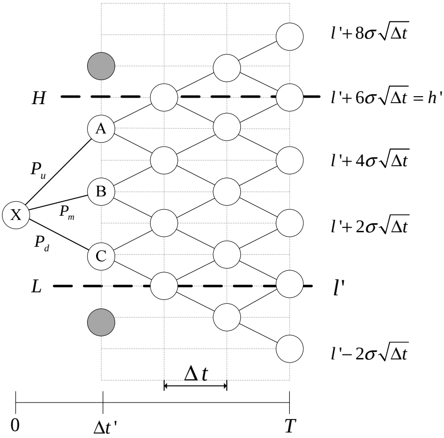


Theorem 1

Given a node X at time t and a CRR lattice with the length of each time step equal to Δt beginning at time $t + \Delta t'$, there is a valid trinomial structure from the node X whose $s(X)$-log-price of the central node B lies in the interval $[\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t})$. Furthermore, the valid branching probabilities can be found by matching the mean and variance of the $s(X)$-log-price of $S_{t+\Delta t'}$.

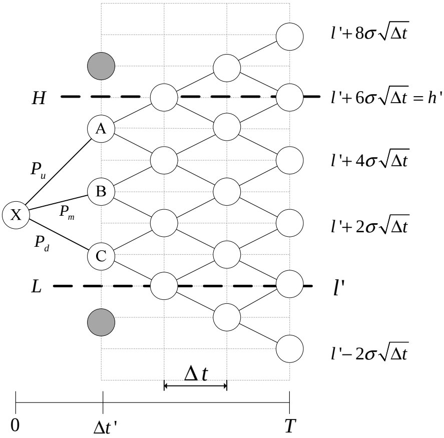
Adjusting the Diffusion Part of the Lattice

- Select Δt to make $\frac{h'-l'}{2\sigma\sqrt{\Delta t}}$ be an integer.
- $\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1\right) \Delta t.$



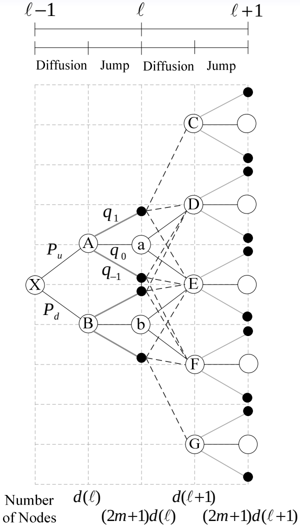
Adjusting the Diffusion Part of the Lattice

- Lay out the grid from barrier L upward.
- Automatically, barrier H coincides with one level of nodes.
- Obtain P_u, P_m, P_d by Theorem 1 (p. 22).

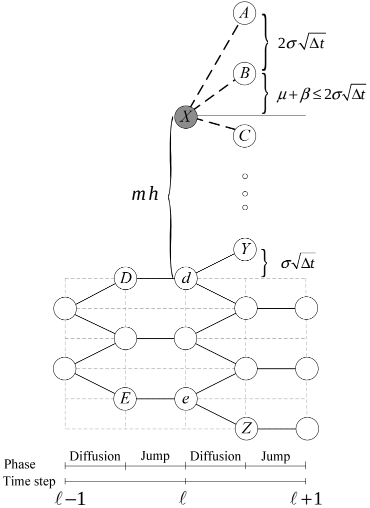


Dealing with Jump Nodes

- Two phases: the diffusion phase and the jump phase.
- The node count of our lattice is $O(n^{2.5})$.



Complexity Analysis



- Node X is the highest jump node at time step ℓ.
- Node d is the highest diffusion node at time step ℓ.
- The distance between node A and Y is $< mh + 3\sigma\sqrt{\Delta t}$.
- At each time step, the number of extra diffusion nodes is at most $2\left\lceil \frac{mh+3\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} \right\rceil$ (such as nodes A, B, C on the left).

Complexity Analysis

- Define $d(\ell)$ as the number of diffusion nodes at time step ℓ.
- $d(\ell)$ satisfies the following recurrence relation:

$$\begin{aligned} d(\ell+1) &= d(\ell) + 2\left\lceil \frac{mh}{2\sigma\sqrt{T/n}} + 1.5 \right\rceil + 1, \\ &= \qquad \vdots \\ &= d(1) + 2 \times (\ell)\left\lceil \frac{mh}{2\sigma\sqrt{T/n}} + 1.5 \right\rceil + \ell, \\ &= O(n^{1.5}), \end{aligned}$$

where $d(1) = 2$, $d(0) = 1$, and $\ell \leq n$.

- Consequently, the node count of the whole lattice is $(2m+1)\sum_{\ell=0}^{n} d(\ell) = O(n^{2.5})$.

Time Complexity

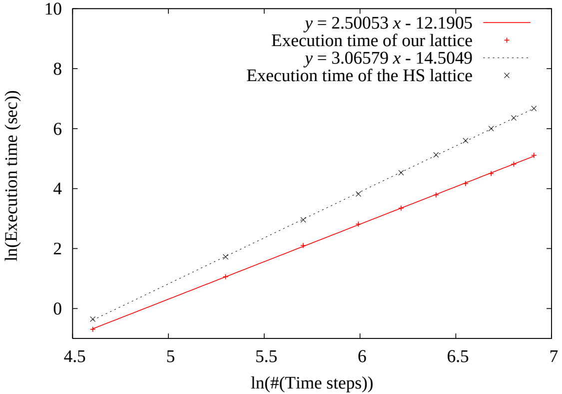


Figure: time complexity.

Vanilla Options

	European puts			
Strike	Merton	Our Model	H&S	Amin
30	2.172	2.194	2.189	1.837
35	3.810	3.793	3.788	3.553
40	5.980	6.002	6.004	5.783
45	8.650	8.630	8.638	8.501
50	11.756	11.773	11.787	11.646
	MAE	**0.020**	0.021	0.210
	RMSE	**0.020**	0.022	0.224
	American puts			
Strike	Benchmark	Our Model	H&S	Amin
30	2.279	2.278	2.272	2.057
35	3.972	3.971	3.965	3.878
40	6.308	6.313	6.313	6.249
45	9.163	9.164	9.165	9.157
50	12.560	12.562	12.566	12.559
	MAE	**0.002**	0.005	0.077
	RMSE	**0.002**	0.006	0.111

Table: Pricing European Puts and American Puts.

Vanilla Options

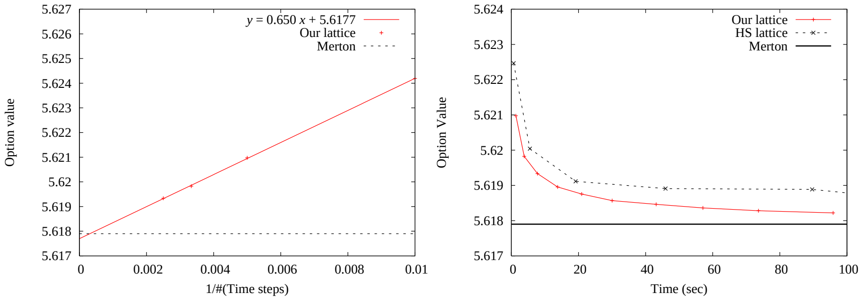


Figure: Converge Property.

Barrier Options

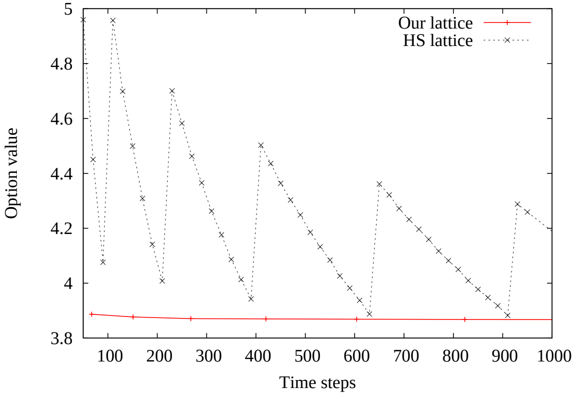


Figure: Pricing a Single-Barrier Call Option.

Barrier Options

Time Steps	European down-and-out and up-and-out call: Simulated Value (+/− 95% bounds)	Our Model	H&S	Percent errors: Our Model	H&S
100		2.593	2.653	−0.103	2.211
401	2.595	2.598	2.635	0.097	1.518
701	(2.584, 2.606)	2.599	2.621	0.125	0.989
1002		2.599	2.630	0.146	1.356

Table: Pricing a Double-Barrier Call Option.

Conclusion

- This talk presents a novel, accurate, and efficient lattice model to price a huge variety of derivatives whose underlying asset follows the jump-diffusion process.
    - It is the first attempt to reduce the time complexity of the lattice model for the jump-diffusion process to $O(n^{2.5})$.
    - In contrast, that of previous work is $O(n^3)$.
    - With the adjustable structure to fit derivatives' specifications, our lattice model make the pricing results converge smoothly.
- According to the numerical results, our lattice model is superior to the existing methods in terms of accuracy, speed, and generality.