An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process

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April 3, 2009
An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process
A derivative product is a financial instrument whose payoff is based on other underlying assets such as stocks.

Pricing it is equivalent to computing its expected payoff under a suitable probability measure.

Most derivatives have no analytical formulas.

So they must be priced by numerical methods like the lattice model.
However, the nonlinearity error may cause the pricing results to converge slowly.

It may even cause the pricing results to oscillate significantly.\footnote{Figlewski and Gao (1999).}

The goals are numerical accuracy and speed.
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Models

- Lognormal diffusion process has been widely used to model the underlying asset’s price dynamics.\(^2\)

- Unfortunately, the lognormal diffusion process is incapable of capturing empirical stock price behaviors.\(^3\)

- Many alternative processes, such as GARCH process, jump-diffusion process, have been proposed.

\(^2\)Black and Scholes (1973).
Related Work

- **Amin (1993)**
  - He approximates the jump-diffusion process by a multinomial lattice.
  - Huge numbers of branches at each node make the lattice inefficient.

- **Hilliard and Schwartz (2005)**
  - They match the first local moments of the lognormal jumps.
  - Their lattice lacks the flexibility to suit derivatives’ specifications.
Main Results

- This talk proposes an efficient lattice model for the jump-diffusion process.
- The time complexity of our lattice is $O(n^{2.5})$.
- Our lattice is adjusted to suit the derivatives’ specification so that the price oscillation problem can be significantly suppressed.
Define $S_t$ as the stock price at time $t$.

Merton's jump-diffusion model assumes that the stock price process can be expressed as

$$S_t = S_0 e^{(r - \lambda \bar{k} - 0.5\sigma^2) t + \sigma z(t) U(n(t))}.$$  \hspace{1cm} (1)

- $z(t)$ denotes a standard Brownian motion.
- $r$ denotes the risk-free rate.
- $\sigma$ denotes the volatility of the diffusion component of the stock price process.
- $U(n(t)) = \prod_{i=0}^{n(t)} (1 + k_i)$ and $k_0 = 0$. 

Jump events are governed by the Poisson process $n(t)$ with jump intensity $\lambda$.

The random jump magnitude $k_i$ ($i > 0$) follows the equation:

$$\ln(1 + k_i) \sim N(\gamma, \delta),$$

where $E(k_i) \equiv \bar{k} = e^{\gamma+0.5\delta^2} - 1$. 

Jump-Diffusion Process

Hilliard and Schwartz decompose $S_0$-log-price of $S_t$ into the diffusion component and the jump component by rewriting Eq. (1) as follows:

$$V_t \equiv \ln \left( \frac{S_t}{S_0} \right) = X_t + Y_t,$$

- The diffusion component
  $$X_t \equiv (r - \lambda \bar{k} - 0.5 \sigma^2) t + \sigma z(t)$$
  is a Brownian motion.
- The jump component
  $$Y_t \equiv \sum_{i=0}^{n(t)} \ln (1 + k_i)$$
  is normal under Poisson compounding.
An option is a financial instrument.

It represents a right to buy the stock for a price (the exercise price $X$) at the maturity date $T$.

Options are essential to speculation and the management of financial risk.$^4$

The payoff of a European-style vanilla option at the maturity date $T$ is $\max(S_T - X, 0)$.

The payoff of an American-style vanilla option at time $t$ ($0 \leq t \leq T$) is $S_t - X$.

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$^4$Hull (2002).
A barrier option is similar to a vanilla option.

But the payoff of a barrier option depends on whether the underlying stock’s price path ever touches the barrier(s).

Such options are very popular in the financial market.
Barrier Options

Price

$H$

$S$

Barrier hit

Time

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The size of one time step is $\Delta t = T/n$.

$u, d, P_u, P_d$:

- Match the mean and variance of the stock return.
- $ud = 1$.
- $P_u + P_d = 1$. 

An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process
Diffusion part ($X_t$)
- Match mean and variance of $X_{\Delta t}$.
- Obtain $P_u$ and $P_d$.

Jump part ($Y_t$)
- Match the first $2m$ local moments of $Y_{\Delta t}$.
- Obtain $q_j$ ($j = 0, \pm 1, \pm 2, \ldots, \pm m$).

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Hilliard and Schwartz’s Lattice

- The stock price of the node at time step $i$ is $S_{0}^{V_{i}}\Delta t$.
- Motivated by the decomposition in Eq. (11), $V_{(i+1)\Delta t}$ can be represented by
  \[ V_{(i+1)\Delta t} = V_{i\Delta t} + c\sigma \sqrt{\Delta t} + jh, \quad j = 0, \pm 1, \pm 2, \ldots, \pm m. \]
  - $c \in \{-1, 1\}$ denotes the upward or the downward movement of the stock price driven by the diffusion component.
  - $j$ denotes the number of positions above or below the node $V_{i\Delta t} + c\sigma \sqrt{\Delta t}$.
    - The magnitude of the basic jump unit is set to $h = \sqrt{\gamma^2 + \delta^2}$.
- The node count of the lattice is $O(n^3)$. 

An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process
Define $F(V_{i\Delta t}, i)$ as the option value (stock price is $S_0 e^{V_{i\Delta t}}$).

European-style options

$$F(V_{i\Delta t}, i) = e^{-r\Delta t} \sum_{j=-m}^{m} F(V_{i\Delta t} + \sigma \sqrt{\Delta t} + jh, (i + 1) \Delta t) P_u q_j + \sum_{j=-m}^{m} e^{-r\Delta t} F(V_{i\Delta t} - \sigma \sqrt{\Delta t} + jh, (i + 1) \Delta t) P_d q_j.$$
Our Lattice

- **Diffusion part \( (X_t) \)**
  - Follow a CRR structure.
  - Follow a trinomial structure.

- **Jump part \( (Y_t) \)**
  - Match the first \( 2m \) local moments of \( Y_{\Delta t} \).
  - Obtain \( q_j \)
    
    \[
    j = 0, \pm 1, \pm 2, \ldots, \pm m
    \]
Price Oscillation Problem

- Price oscillation problem is mainly due to the nonlinearity error.
  - Introduced by the nonlinearity of the option value function.
- The solution of the nonlinearity error:
  - Making price level of the lattice coincide with the location where the option value function is highly nonlinear.
Trinomial Structure

The branching probabilities for the node $X$

\[
\beta \equiv \hat{\mu} - \mu, \\
\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}, \\
\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}, \\
\hat{\mu} \equiv \ln \left( \frac{s(B)}{s(X)} \right).
\]
The branching probabilities for the node $X$

\[ P_u \alpha + P_m \beta + P_d \gamma = 0, \]
\[ P_u (\alpha)^2 + P_m (\beta)^2 + P_d (\gamma)^2 = \text{Var}, \]
\[ P_u + P_m + P_d = 1. \]
Theorem 1

Given a node $X$ at time $t$ and a CRR lattice with the length of each time step equal to $\Delta t$ beginning at time $t + \Delta t'$, there is a valid trinomial structure from the node $X$ whose $s(X)$-log-price of the central node $B$ lies in the interval $[\mu - \sigma \sqrt{\Delta t}, \mu + \sigma \sqrt{\Delta t}]$. Furthermore, the valid branching probabilities can be found by matching the mean and variance of the $s(X)$-log-price of $S_{t+\Delta t'}$. 
Adjusting the Diffusion Part of the Lattice

- Select $\Delta t'$ to make $\frac{h' - l'}{2\sigma \sqrt{\Delta t}}$ be an integer.
- $\Delta t' = T - (\lfloor \frac{T}{\Delta t} \rfloor - 1) \Delta t$. 

![Lattice Diagram](image)
Adjusting the Diffusion Part of the Lattice

- Lay out the grid from barrier $L$ upward.
- Automatically, barrier $H$ coincides with one level of nodes.
- Obtain $P_u$, $P_m$, $P_d$ by Theorem 1 (p. 22).
Dealing with Jump Nodes

- Two phases: the diffusion phase and the jump phase.
- The node count of our lattice is $O(n^{2.5})$. 
Node $X$ is the highest jump node at time step $\ell$.

Node $d$ is the highest diffusion node at time step $\ell$.

The distance between node $A$ and $Y$ is $< mh + 3\sigma \sqrt{\Delta t}$.

At each time step, the number of extra diffusion nodes is at most $2 \left\lceil \frac{mh+3\sigma \sqrt{\Delta t}}{2\sigma \sqrt{\Delta t}} \right\rceil$ (such as nodes $A$, $B$, $C$ on the left).
Complexity Analysis

- Define $d(\ell)$ as the number of diffusion nodes at time step $\ell$.
- $d(\ell)$ satisfies the following recurrence relation:

$$d(\ell + 1) = d(\ell) + 2 \left\lceil \frac{mh}{2\sigma \sqrt{T/n}} + 1.5 \right\rceil + 1,$$

$$= \cdots$$

$$= d(1) + 2 \times (\ell) \left\lceil \frac{mh}{2\sigma \sqrt{T/n}} + 1.5 \right\rceil + \ell,$$

$$= O(n^{1.5}),$$

where $d(1) = 2$, $d(0) = 1$, and $\ell \leq n$.

- Consequently, the node count of the whole lattice is

$$(2m + 1) \sum_{\ell=0}^{n} d(\ell) = O(n^{2.5}).$$
Figure: time complexity.
### Vanilla Options

#### European puts

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<th>Merton</th>
<th>Our Model</th>
<th>H&amp;S</th>
<th>Amin</th>
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#### American puts

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**Table:** Pricing European Puts and American Puts.
Vanilla Options

y = 0.650x + 5.6177

Option value vs. \(1/#(\text{Time steps})\)

Option Value vs. Time (sec)

Figure: Converge Property.
Barrier Options

Figure: Pricing a Single-Barrier Call Option.
## Numerical Results

### Table: Pricing a Double-Barrier Call Option.

<table>
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<th>Time Steps</th>
<th>Simulated Value (+/− 95% bounds)</th>
<th>Our Model</th>
<th>H&amp;S</th>
<th>Percent errors</th>
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An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process
This talk presents a novel, accurate, and efficient lattice model to price a huge variety of derivatives whose underlying asset follows the jump-diffusion process.

- It is the first attempt to reduce the time complexity of the lattice model for the jump-diffusion process to $O(n^{2.5})$.
- In contrast, that of previous work is $O(n^3)$.
- With the adjustable structure to fit derivatives' specifications, our lattice model makes the pricing results converge smoothly.

According to the numerical results, our lattice model is superior to the existing methods in terms of accuracy, speed, and generality.