Comonotonicity and Multiasset Option Pricing

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Main aims of this contribution

- Part I: Optimal arbitrage free static hedging strategies for basket options and new measure of lack of comonotonic dependence in correlated assets: Market Implied Comonotonicity Gap (building on work by Hobson, Laurence and Wang)
- Part II: Extension to generalized spread options.

Basket Options

The Payoff of a basket call option:

$$\psi(S_1, \cdots, S_n) = \left(\sum_i w_i S_i - K\right)^+$$

- Price weighted –> $w_i = \frac{1}{I(t_0)}$.
- Capitalization Weighted,
  $w_i = Cst.\frac{\text{nb. } s_i \text{ shares outstanding}}{\text{Total capitalization}}$.
  w_i are readjusted periodically.
- S&P 500, S&P 100, Dow Jones 100.

DJX Index

Symbol	Name	Last Price	Weight
AA	ALCOA, INC.	34.270	2.40%
AIG	AMERICAN INTERNATIONAL GROUP INC	61.030	4.28%
AXP	AMERICAN EXPRESS CO	55.630	3.90%
BA	BOEING CO	53.930	3.78%
C	CITIGROUP	47.070	3.30%
CAT	CATERPILLAR INC.	89.970	6.31%
DD	DU PONT EI DE NEMOURS	44.500	3.12%
DIS	WALT DISNEY CO	26.800	1.88%
GE	GENERAL ELECTRIC CO	36.250	2.54%
GM	GENERAL MOTORS CORP	40.210	2.82%
HD	HOME DEPOT INC	43.220	3.03%
HON	HONEYWELL INTERNATIONAL INC.	36.620	2.57%
HPQ	HEWLETT PACKARD CO	19.340	1.36%
IBM	INTERNATIONAL BUSINESS MACHINES	95.320	6.68%
INTC	INTEL CORP	23.690	1.66%
JNJ	JOHNSON AND JOHNSON	61.000	4.28%
JPM	JP MORGAN CHASE AND CO INC	39.170	2.75%
KO	COCA COLA CO	40.790	2.86%
MCD	MCDONALDS CORP	30.500	2.14%
MMM	3M COMPANY	82.680	5.80%
MO	ALTRIA GROUP INC.	54.740	3.84%
MRK	MERCK AND COMPANY INC	26.450	1.85%
MSFT	MICROSOFT CORP	26.970	1.89%
PFE	PFIZER INC.	27.450	1.92%
PG	PROCTER AND GAMBLE CO	54.600	3.83%

Introducing the GAP

We will now introduce a quantity called "the Gap", or more precisely "Market Implied Comonotonicity Gap" (for short: MICG), with the property that:

- Gap can be monitored over time and used as a tool in a static (or semi-static) dispersion trading strategy.
- When gap is small ("High correlation") compared to it's historical values: basket is overpriced.

  $\Rightarrow$ Sell basket option, buy options on the components.

- When gap is big compared to it's historical values ("Low correlation"): basket is cheap, undervalued.

  $\Rightarrow$ Buy an option on the basket, sell options on the components

Implied Correlation

We will describe MICG and contrast with another well known dispersion trading strategy, so called "implied correlation.

- Implied correlation is the number ρ such that when ρ_{ij} are replaced by ρ gives same implied variance of index:

$$\sigma_I^2 = \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{n} \sigma_i^2 + \rho \sum_{i \neq j} \sigma_i \sigma_j$$

Hence,

$$\rho = \frac{\sigma_I^2 - \sum\limits_{i=1}^{n} \sigma_i^2}{\sum\limits_{i \neq j} \sigma_i \sigma_j}$$

Implied correlation 2

- But

$$\sigma_I = \sigma_I(K^{bask}),$$

so which strikes $K_i, i = 1, \cdots, n$ should we use to select $\sigma_i = \sigma_i(K_i), i = 1, \cdots, n$ in the above formula?
Wide spread practice:

$$K^{bask} \text{ ATM, then choose } \quad K_i \quad \text{ ATM}$$

But what if K^{bask} is out of or in the money? Or even for ATM in what sense is choice of ATM K_i optimal?

- In contrast MICG gives means of selecting optimal strikes.

A new measure of correlation

- Plan: We will recall the definition of comonotonicity and will illustrate the difference between perfect positive correlation and co-monotonicity.
- We introduce as a measure of lack of comonotonicity of components in a basket product:

$$\boxed{\text{Gap} = \mathcal{C} - \mathcal{M}}$$

  - $\mathcal{C}$: the market implied comonotonic price
  - $\mathcal{M}$: true market price

Co-monotonicity

Recall the definition of co-monotonicity:

A random vector $(X_1, X_2, \cdots X_n)$ is said to be co-monotonic if there exists a uniformly distributed random variable U such that

$$U \sim \text{Uniform}(0,1)$$

$$(X_1, X_2, \cdots, X_n) \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right),$$

where $F_{X_i}(x)$ is the distribution function of X_i.

Perfect positive correlation $\neq$ co-monotonicity

Difference between perfect positive correlation and co-mononotonicity. Tchen, Dhaene-Denuit's theorem, concerning the relation of linear correlation with comonotonicity:

Theorem 1 *If (X_1, X_2) is a random vector with given margins F_{X_1}, F_{X_2} and let ρ be the Pearson (i.e., linear, standard) correlation coefficient, then we have*

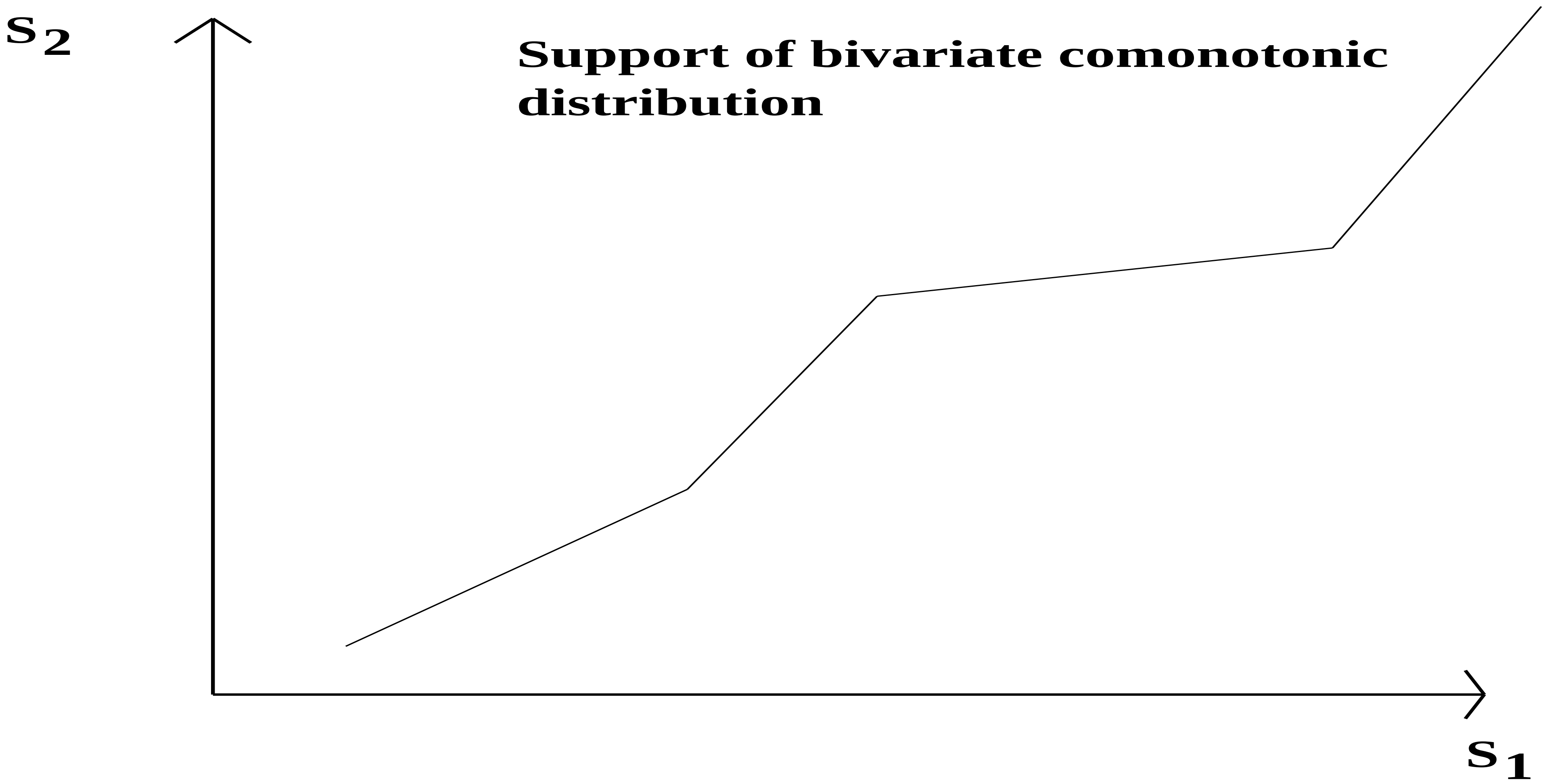
$$\rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U)) \leq \rho(X_1, Y_1) \leq \rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)),$$

where U is a uniformly distributed random variable.

In words:

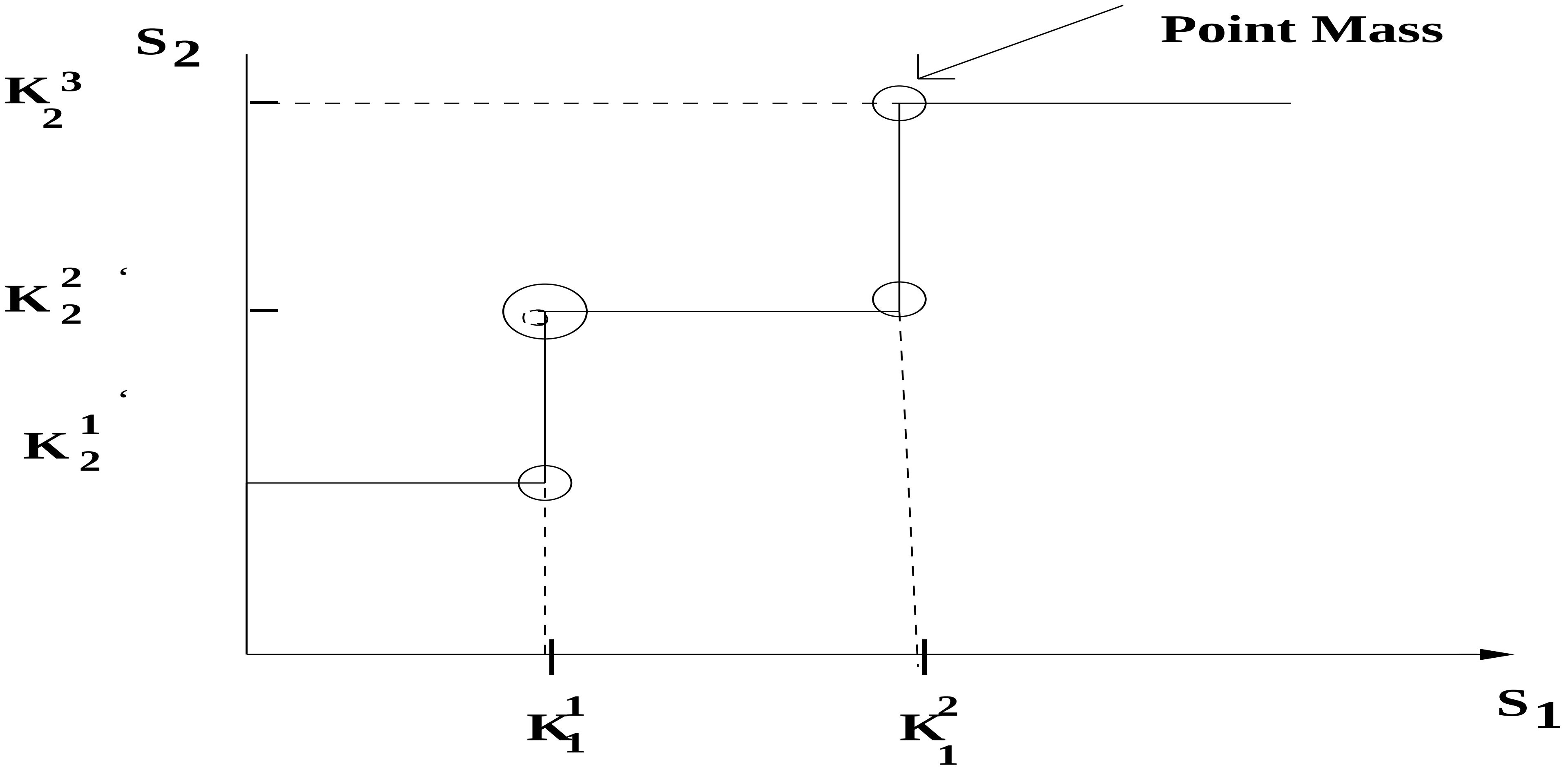
- Largest value of the correlation for a random vector (X_1, X_2) with given marginals is attained for comonotonic random variables, but is generally not equal to 1 unless they have a linear dependence with positive slope ($X_2 = aX_1 + b, a > 0$).
- Minimal value of the correlation for a random vector $(X_, X_2)$ with given marginals is attained for antimonotonic random variables, but is generally not equal to -1.

Continuous co-monotonic



S_2 is non−decreasing function of S_1

Comonotonic Distribution: purely atomic, with jumps



A comonotonic distribution with jumps

How to determine $\mathcal{C}$?

So, given a basket options with payoff

$$\left(\sum w_i S_i - K\right)^+$$

how do we determine the comonotonic price?

- ANSWER: If we knew with certainty the marginals F_{S_i} of the individual assets S_i in the basket, the the procedure would be:
- First determine the joint probability distribution for the stocks in the basket via

$$P\left(S_1 \leq x_1, S_2 \leq x_2, \cdots, S_n \leq x_n\right)$$
$$= C^U_{\text{Fréchet}}\left(F_{S_1}(x_1), F_{S_2}(x_2), \ldots, F_{S_n}(x_n)\right)$$

where

$$\boxed{C^U_{\text{Fréchet}}(y_1, y_2, \cdots, y_n) = \min\left(y_1, y_2, \ldots, y_n\right) \quad \text{upper Fréchet bound}}$$

The Gap II

- Second: Determine the density of joint prob. distribution of the basket via

$$p(x_1, x_2, \cdots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} \left[P \left(S_1 \leq x_1, S_2 \leq x_2, \cdots, S_n \leq x_n \right) \right]$$

- Third:

$$BasketPrice = \int_{\Re_n^+} \left(\sum_{i=1}^{n} S_i - K \right)^+ p(S_1, S_2, \ldots, S_n) dS_1 \ldots dS_n$$

Where do marginals come from?

- Recall Breeden-Litzenberger theorem (Journal of Finance, 1978):

  Theorem 2 *Let $C(S,t,K,T)$ be call prices corresponding at time t and given that the spot price is at S, for a call option struck at K and expiring at T, assuming a continuum of strikes is traded.*
  Then

$$\frac{\partial^2}{\partial K^2} C(S,t,K,T) = e^{-r(T-t}p(S,t,K,T) \quad \text{where } p \text{ is the transition probability}$$

$$\Rightarrow \text{marginal distribution function of } S \text{ i.e. } F_S(s) \text{ is therefore known}$$

- In reality, the market provides us only with a finite number of strikes for each expiry and for each stock $S = S_i, i = 1, \cdots, n$. So how do we fill in Call price functions for each asset for all strikes? Answer related (but only very partially explained) by work on distribution free bounds for one asset, of which we now give a reminder:

A typical Component Option, Procter& Gamble

May, 2004 | July, 2004 | October, 2004 | January, 2005 | January, 2006

PROCTER & GAMBLE CO 105.97 ▼ -0.15 -0.1414% 105.91 106.37 2,727,800

Calls							Strike	Puts						
Symbol	Last	Chg	Bid	Ask	Vol	Int	Price	Symbol	Last	Chg	Bid	Ask	Vol	Int
PG EM	41.50	0.00	40.80	41.10	0	15	65	PG QM	0.00	0.00	0.00	0.05	0	.
PG EN	36.50	0.00	35.80	36.10	0	65	70	PG QN	0.00	0.00	0.00	0.05	0	.
PG EO	31.50	0.00	30.90	31.10	0	15	75	PG QO	0.00	0.00	0.00	0.05	0	.
PG EP	26.00	0.00	25.90	26.10	0	.	80	PG QP	0.05	0.00	0.00	0.05	0	20
PG EQ	21.00	0.00	20.90	21.10	0	40	85	PG QQ	0.00	0.00	0.00	0.05	0	.
PG ER	16.00	0.00	15.90	16.10	0	58	90	PG QR	0.10	0.00	0.00	0.10	0	90
PG ES	11.30	0.00	10.90	11.10	0	204	95	PG QS	0.20	0.00	0.10	0.20	0	173
PG ET	6.00	-0.10	6.00	6.20	132	229	100	PG QT	0.45	-0.05	0.45	0.50	158	3,229
PG EA	2.35	-0.35	2.30	2.35	1,287	3,049	105	PG QA	1.60	0.00	1.70	1.75	680	2,065
PG EB	0.50	0.00	0.45	0.50	193	2,921	110	PG QB	4.70	-0.50	4.80	5.00	305	1,639
PG EC	0.05	0.00	0.05	0.10	15	258	115	PG QC	10.10	0.00	9.50	9.70	0	64
PG ED	0.00	0.00	0.00	0.05	0	.	120	PG QD	14.40	0.00	14.40	14.70	0	75
PG EE	0.00	0.00	0.00	0.05	0	.	125	PG QE	19.70	0.00	19.40	19.70	0	138
PG EF	0.00	0.00	0.00	0.05	0	.	130	PG QF	24.10	-0.60	24.40	24.70	12	163

The "Market Implied" co-monotonicity gap

- The market only gives us *partial information* about the marginals through the prices of traded options with various traded strikes $K_1^{(i)}, K_2^{(i)}, \cdots, K_{J(i)}^{(i)}$ for stock S_i at a given maturity t.
- Let UB be the upper bound for basket option, given only this partial information, then

$$\boxed{\text{Market implied comonotonicity Gap} = \text{UB} - \text{traded Market Price}}$$

- Fundamental: Given a basket option on n assets, there is a portfolio $\mathcal{P}$ of $n+1$ options on components, such that

$$\text{UB} = \text{Market Price of } \mathcal{P}$$

Below we will discuss how to determine the upper bound UB.

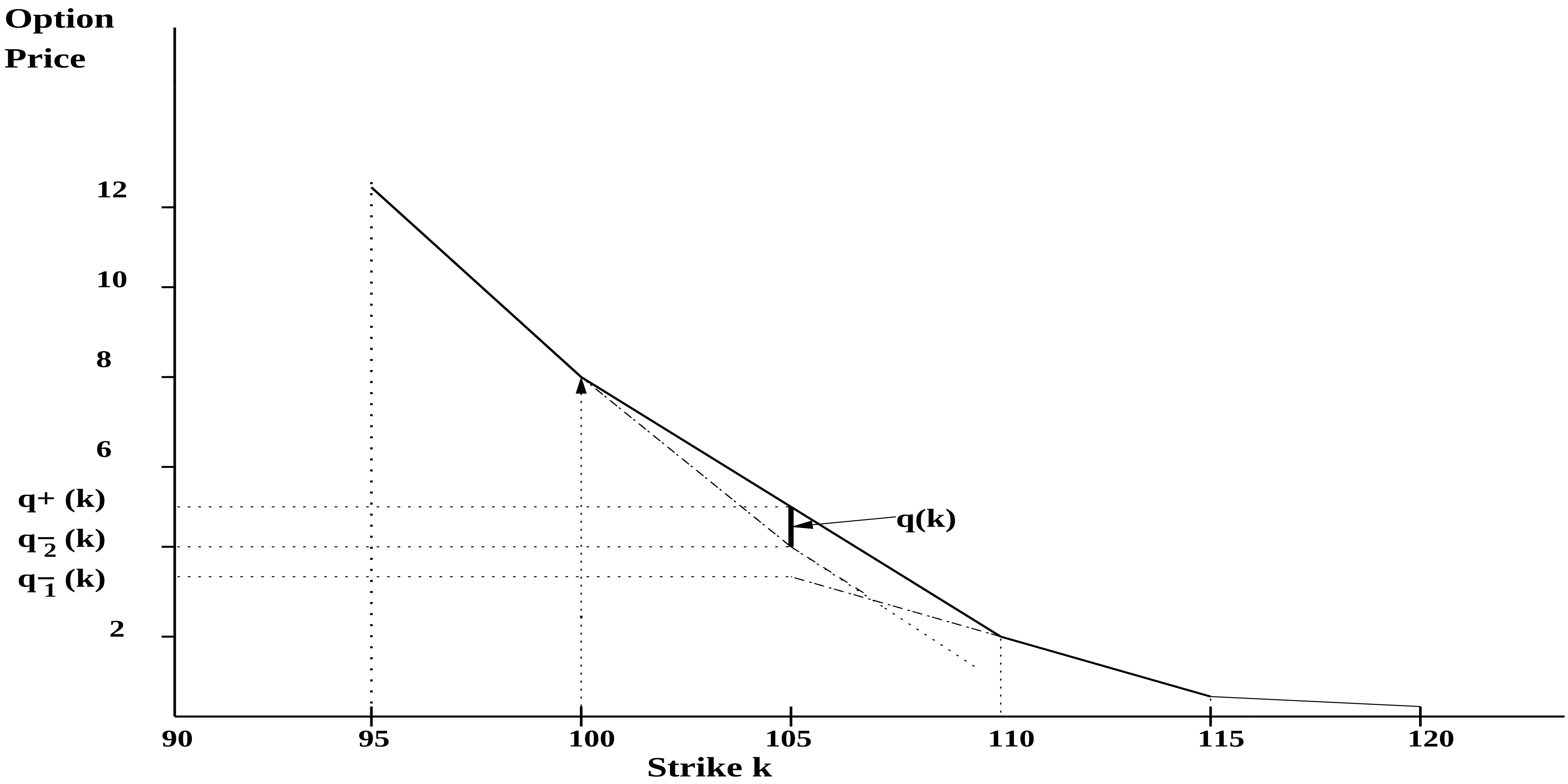
Recent Work on Model Independent Option B

Bertsimas and Popescu, 2003, use a LP approach to derive bounds on assets under a variety of constraints. Here is one of their results:

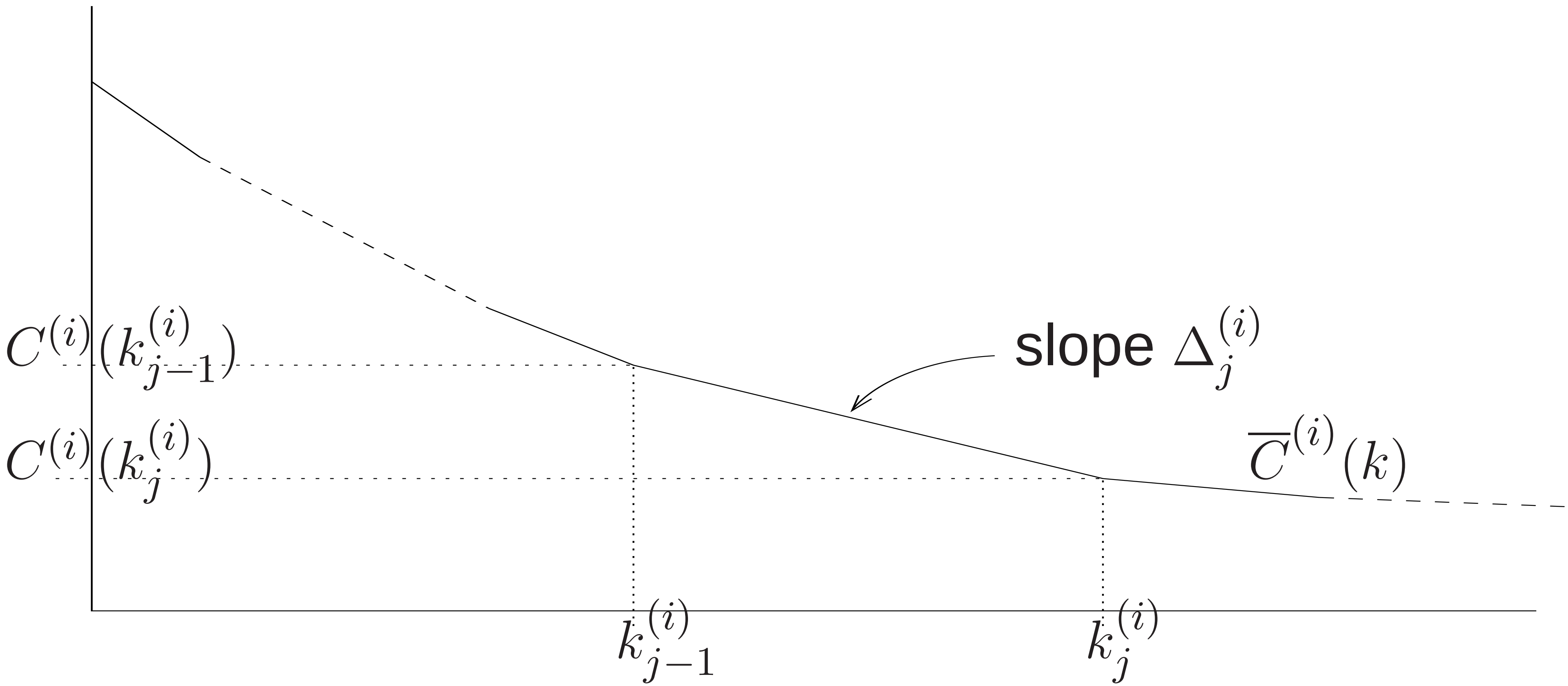
Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq .. \leq K_n$ on a stock X, the range of all possible prices for a call option with strike K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[C^-(K), C^+(K)]$ where

$$
\begin{aligned}
& C^-(K) \\
& = \max \left(C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}}, \right. \\
& \qquad \left. C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}} \right) \quad \text{lower bounds} \\
& C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j} \quad \text{upper bounds}
\end{aligned}
$$

Bertsimas-Popescu



Linear interpolation



The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the slope of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$. This graph provides one of many ways of filling in the missing strikes. But it turns out to be the **fundamental interpolation**, in the case of the upper bound.

Co-monotonic copula & Option Prices

- The marginals corresponding to piecewise linear call prices are discontinuous at every strike price and constant between strike prices.
  Because:
-

$$\frac{\partial^2 C^{(i)}}{\partial K^2} = \text{density}$$

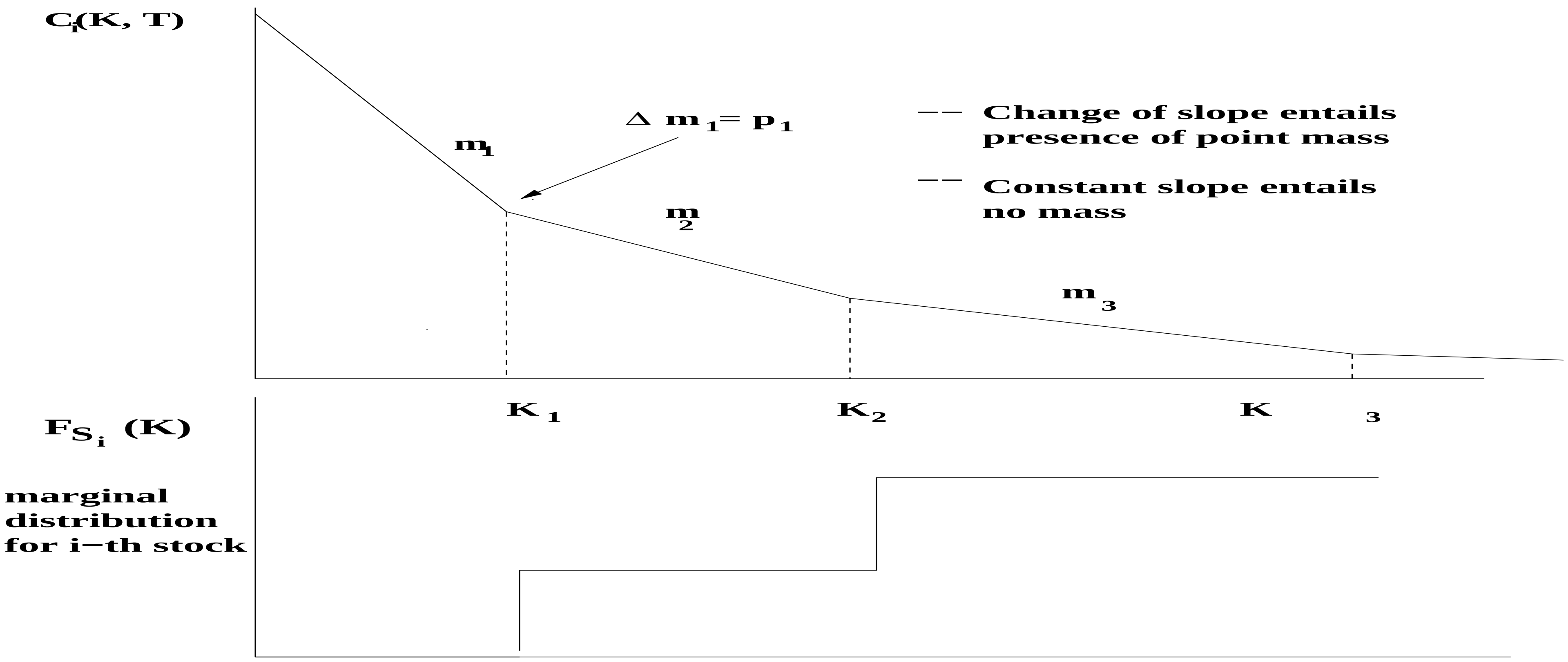
and because our call price functions are piecewise linear *between* two strikes so

$$\frac{\partial^2 C}{\partial K^2} = 0, \quad K_i^j \leq K \leq K_i^{j+1}$$

$$\frac{\partial^2 C}{\partial K^2} = \delta(K_i^j) \times \left(\text{change of slope at } K_i^j\right),$$

This is illustrated in following slide:

Underlying assets have jumps and regions with no mass



The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the slope of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$.

Optimizer

- Now the market implied co-monotonic optimizer $(\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n)$ is a random variable which is distributed like the vector random variable

$$\Big((F^M_{S_1})^{-1}(U), (F^M_{S_2})^{-1}(U), \ldots, (F^M_{S_n})^{-1}(U)\Big)$$

  where $F^M_{S_i}, i = 1, \cdots, n$ are the market implied marginals with point masses at the strikes.

- It can be shown (Laurence and Wang (2004, 2005) and Hobson, Laurence and Wang (2005)) that the market implied co-monotonic optimizer is a solution of optimization problem on next slide:

Optimization - primal

Constrained optimization problem. Determine

$$\sup_{\mu} \int \left(\sum_i w_i S_i - K \right)^+ \mu(dS)$$

subject to

$$\int (S_i - k_j^{(i)})^+ \mu(dS) = C^{(i)}(k_j^{(i)}), \text{ for } i = 1, \ldots, n, j = 1, \ldots, J^{(i)}$$

$$\int \mu(dS) = 1$$

Optimization - dual

Dual problem

$$\inf_{\nu,\psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_j^{(i)})\nu_i^j + \psi$$

subject to

$$\left(\sum_i w_i S_i - K \right)^+ \leq \sum_{i,j} \left(S_i - k_j^{(i)} \right)^+ \nu_i^j + \psi \qquad \text{(*)}$$

$\nu_j^i \in \mathbb{R}$, for $i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)}$

$\psi \in \mathbb{R}$

(*) is the super-replication condition

Here ψ is cash component and ν_j^i is number of options with strike k_j^i in hedging portfolio.

Finite market - Using all traded options

- **Preliminaries** For simplicity of exposition assume all slopes $\left.\frac{\partial C^{(i)}(u)}{\partial u}\right|_{u=k_j^{(i)}}$ are different as i and j vary. Let $I_n = \{1, 2, \cdots, n\}$ where n is the number of assets.
- There is a privileged index $\hat{i} \in I_n$ such that:

  For any model which is consistent with the observed call prices $C^{(i)}(K_j)$, the price $B(K)$ for the basket option is bounded above by $\overline{\mathcal{B}}_F(K)$, where
- Case I: $\sum_i w_i k^{(i)}_{J(i)} > K$:

$$\overline{\mathcal{B}}_F(K) = \sum_{i \in I_n \setminus \hat{i}} w_i C^{(i)} \left(k^{(i)}_{\overline{j}(i)} \right) + w_{\hat{i}} \left\{ (1 - \theta^*_{\hat{i}}) C^{(\hat{i})} \left(K^{(\hat{i})}_{\overline{j}(\hat{i})-1} \right) + \theta^*_{\hat{i}} C^{(\hat{i})} \left(k^{(\hat{i})}_{\overline{j}(\hat{i})} \right) \right\}$$

- $\theta^*_{\hat{i}}$ is defined as $\theta^*_{\hat{i}} = \frac{\overline{\lambda}^*_{\hat{i}} - \overline{\lambda}^-_{\hat{i}}(\phi^*)}{\overline{\lambda}^+_{\hat{i}}(\phi^*) - \overline{\lambda}^-_{\hat{i}}(\phi^*)} = \frac{(K\overline{\lambda}^*_{\hat{i}}/w_{\hat{i}}) - k^{(i)}_{\overline{j}(\hat{i})-1}}{k^{(i)}_{\overline{j}(\hat{i})} - k^{(i)}_{\overline{j}(\hat{i})-1}}, \overline{\lambda}^*_{\hat{i}} \in [k^{(i)}_{\overline{j}(\hat{i})-1}, k^{(i)}_{\overline{j}(\hat{i})}]$.

Finite market - Result

- Case II: $\sum_i w_i K_{J(i)} \leq K$:

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)} \left(k^{(i)}_{J(i)} \right)$$

****

- Based on experiments with real data, the second case essentially never arises in practice.
- Moreover, the upper bound is optimal in the sense that we can find co-monotonic models which are consistent with the observed call prices and for which the arbitrage-free price for the basket option is arbitrarily close to $\overline{\mathcal{B}}_F(K)$.
- So where's the beef in Case I?
- All the beef in fleshing out the estimate in the first case is in determining the special index $\hat{i}$ and the indices $j(i), i = 1 \cdots, n$.

How to find which options to choose?

- Possible to show that there is No cash component ψ in the optimal portfolio. So can consider super-replicating portfolios consisting entirely of options with various strikes (some of which may have strike zero).
- The upper bound is available in quasi-closed form, meaning there is a simple algorithm to determine the solution, modulo a slope ordering algorithm: Order all slopes of all call price functions and cycle through.
- To get the intuition as to how to proceed, note that if $\sum \lambda_i = 1$ then

$$\left(\sum_i w_i X_M^{(i)} - K\right)^+ \leq \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i K}{w_i}\right)^+, \quad \text{due to Merton}$$

So that

$$C_{\mathcal{B}}(K) \leq \sum_i w_i C^{(i)}\left(\lambda_i K / w_i\right).$$

The λ_i are arbitrary and so $C_{\mathcal{B}}(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)}\left(\lambda_i K / w_i\right).$

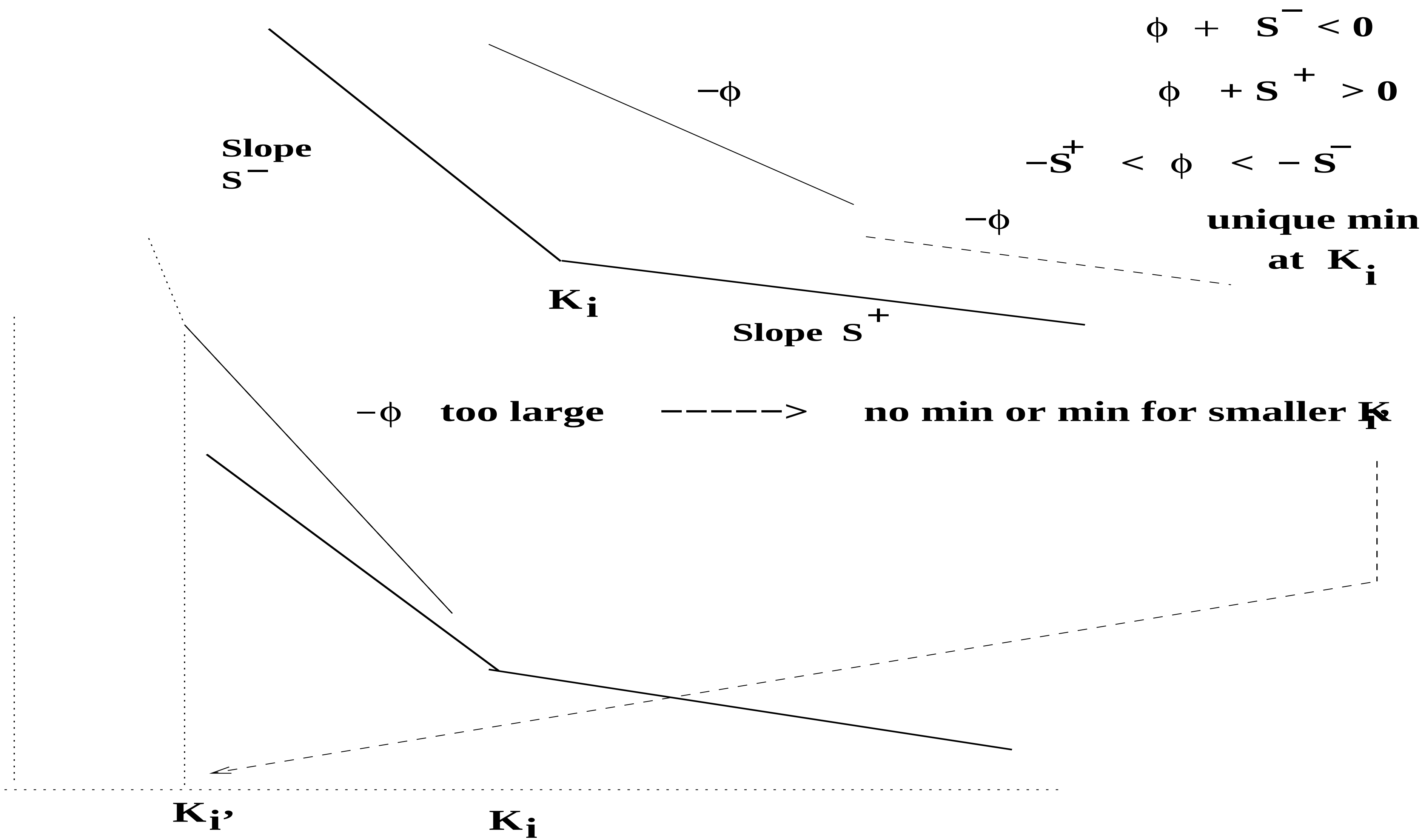
Intuition

- We wish to find the infimum of $\sum_i w_i C^{(i)}(\lambda_i K/w_i)$ over choices λ_i satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$. Define the Lagrangian

$$L(\lambda, \phi) = \sum_i w_i C^{(i)}(\lambda_i K/w_i) + \phi\left(\sum_i \lambda_i - 1\right).$$

- Objective function is convex but only $C^{0,1}$, because each piecewise linear call price functions $C^{(i)}$, is $C^{0,1}$, ie. $\frac{\partial C^i}{\partial K}$ has a jump at each strike $K_i^j, j = 1, \cdot, n_i$.
- Note that objective functional is separable function of 1-dimensional functions.
- Therefore for each fixed Lagrange Multiplier ϕ, the gradient can point in a cone of different directions. In the terminology of convex analysis we have $\phi/\beta K \in \bar{\partial} C^{(i)}(\lambda_i K/w_i)$, where $\bar{\partial}$ is the *subdifferential* of the function $C^{(i)}$. .

Illustration Min



Algorithm

For each ϕ there is either a unique $\lambda(\phi)$ or an interval $[\lambda^-(\phi), \lambda^+(\phi)]$.
Essentially:

- $[\lambda(\phi)^-, \lambda(\phi)^+] \sim [w_i K_i^j / K, w_i K_i^{j+1}/K]$ for some i and j.
- So Algorithm:
  - Order all the slopes of all call price functions. Ie. if 30 assets and 8 non zero strikes , order 240 slopes.

$$\mathcal{S}_1 \leq \mathcal{S}_2 \leq \cdots\cdots \leq \mathcal{S}_{240}$$

  - Now starting with $\phi = \epsilon << 1$ increase ϕ while monitoring the quantity

$$\Lambda(\phi) = \sum \lambda^+(\phi)$$

which starts very large for small ϕ ($\Leftarrow$ large K_i^j) and decreases as $\phi \uparrow$.

  - The first time $\Lambda(\phi)$ crosses 1. STOP! $\mapsto$ Optimal value of $\phi = \phi^*$ has been reached.

Experiment on Real DJX Data: Spot was 99.07

We now illustrate the output on real DJX data.

DJX Strikes	DJX Call Prices	AA	AIG	AXP	BA	C	CAT	DD	DIS	GE
52	47.1	0	0	0	37.5	0	0	0	17.5	25
56	43.1	0	0	0/42.5	37.5	0	0	0	17.5	25
60	39.1	22.5	0	42.5	37.5	0/37.5	0	0	17.5	25
64	35.1	22.5	0	42.5	37.5	37.5	0/60	0	17.5	25
68	31.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
70	29.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
72	27.1	22.5	0/60	42.5	37.5	37.5	60	0	17.5	25
76	23.1	22.5	60	42.5	37.5	37.5	60	0	17.5	25
80	19.1	22.5	60	42.5	37.5	37.5	60	0/37.5	17.5	25
84	15.2	22.5	60	42.5	37.5	37.5	60	37.5	20	25
88	11.3	22.5	60	42.5	37.5	40	65	37.5	20	27.5
90	9.4	25	60	45	37.5/40	40	65	37.5	20	27.5
92	7.5	25	65	45	40	42.5	70	37.5/40	20	27.5
94	5.8	25	65	47.5	40	42.5	70	40	22.5	27.5
95	4.95	27.5	65	47.5	40	42.5	70	40	22.5	27.5
96	4.15	27.5	65	47.5	40	42.5	70	40	22.5	30
97	3.35	27.5	70	47.5	42.5	42.5	70	40	22.5	30
98	2.725	27.5	70	47.5	42.5	45	70	40	22.5	30
99	2.125	27.5	70	50	42.5	45	75	40/42.5	22.5	30
100	1.6	30	70	50	42.5	45	75	42.5	22.5	30
102	0.775	30	70	50	45	47.5	75	42.5	22.5	30
103	0.5	30	75	50	45	47.5	75/80	42.5	25	30/32.5
104	0.325	32.5	75	50	45	47.5	80	42.5	25	32.5
105	0.15	32.5	75	50	45	47.5	80	42.5	25	32.5
106	0.15	32.5	75	50	45	47.5	80	45	25	32.5
107	0.15	32.5	75	50	45	47.5	80	45	25	32.5

TABLE 4. The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicacting portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In others the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset increases as the strike on the DJX increases.

How good is the Upper Bound? Spot was 99.07

DJX Strikes	DJX Prices	UB Unclean Data	UB Clean Data	BS Price $\rho = 0$	BS Price $\rho = .5$	BS Price $\rho = .75$	BS Price $\rho = .9$	BS Price $\rho = .99$
52	47.10	47.09	47.05	47.14	47.14	47.15	47.10	47.18
56	43.10	43.10	43.09	43.16	43.18	43.17	43.15	43.17
60	39.10	39.11	39.10	39.16	39.18	39.13	39.12	39.14
64	35.10	35.11	34.30	35.16	35.16	35.16	35.20	35.17
68	31.10	31.12	30.83	31.17	31.17	31.22	31.17	31.16
70	29.10	29.13	29.12	29.18	29.19	29.18	29.17	29.11
72	27.10	27.14	27.14	27.19	27.22	27.18	27.13	27.18
76	23.10	23.15	22.38	23.18	23.16	23.18	23.15	23.19
80	19.10	19.18	19.18	19.20	19.18	19.15	19.19	19.22
84	15.20	15.24	14.95	15.21	15.24	15.23	15.18	15.23
88	11.30	11.42	11.42	11.20	11.26	11.25	11.25	11.36
90	9.40	9.61	9.61	9.21	9.28	9.35	9.41	9.44
92	7.50	7.90	7.90	7.21	7.34	7.53	7.67	7.73
94	5.80	6.32	6.32	5.22	5.58	5.83	6.01	6.08
95	4.95	5.57	5.57	4.22	4.79	5.06	5.26	5.34
96	4.15	4.85	4.85	3.22	4.01	4.35	4.54	4.66
97	3.35	4.19	4.19	2.24	3.28	3.69	3.92	4.01
98	2.73	3.58	3.58	1.35	2.70	3.12	3.34	3.44
99	2.13	3.02	3.02	0.67	2.16	2.58	2.75	2.96
100	1.60	2.53	2.53	0.25	1.69	2.10	2.33	2.43
102	0.78	1.73	1.73	0.01	0.99	1.37	1.55	1.71
103	0.50	1.42	1.42	0.00	0.71	1.05	1.26	1.36
104	0.33	1.16	1.16	0.00	0.52	0.82	1.02	1.13
105	0.15	0.95	0.95	0.00	0.36	0.63	0.79	0.89
106	0.15	0.75	0.75	0.00	0.25	0.48	0.60	0.70
107	0.15	0.59	0.59	0.00	0.16	0.35	0.48	0.53

Spread option case

- The methodology for basket options can also be applied to generalized spread options.
- The payoff ψ of the generalized spread options

$$\psi(S_1, \cdots, S_n) = \left(\sum_{i=1}^{n} w_i S_i - K \right)^+$$

where the weights w_i are constants of arbitrary sign.

- Examples contain heating oil crack spread ($(42 \times [HO] - [CO] - K)^+$), 3:2:1 crack spread ($\left(42 \times \frac{2}{3}[UG] + 42 \times \frac{1}{3}[HO] - [CO] - K\right)$)
Note: 1 barrel = 42 gallons

Anti-monotonicity instead

- Let us group the payoff function for the generalized spread option as

$$\psi(S_1, \cdots, S_n) = \left(\sum_{i \in I^+} w_i S_i - \sum_{i \in I^-} |w_i| S_i - K \right)^+$$

  where I^+ denotes the set of indices with positive weights and I^- the negative weights.
- The upper bound is attained when
  - Assets indexed in I^+ are co-monotonic to one another.
  - Assets indexed in I^- are also co-monotonic to one another.
  - Any asset in I^+ is anti-monotonic to every asset in I^-.
- Special case: $\psi(S_1, S_2) = (S_1 - S_2 - K)^+$
  Upper bound is attained when S_1 and S_2 are anti-monotonic.

Anti-monotonicity

Recall the definition of anti-monotonicity:

A two dimensional random vector (X_1, X_2) is said to be anti-monotonic if there exists a uniformly distributed random variable U such that

$$U \sim \text{Uniform}(0,1)$$

$$(X_1, X_2) \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U)\right),$$

where $F_{X_i}(x)$ is the distribution function of X_i.

Spread option

Therefore, for the generalized spread options with payoff

$$\left(\sum_{i\in I^+} w_i S_i - \sum_{i\in I^-} |w_i| S_i - K\right)^+,$$

the upper bound is attained if there exists a uniformly distributed random variable $U \sim \text{Uniform}(0,1)$ such that

- $S_i \overset{d}{=} F_{X_i}^{-1}(U)$ for $i \in I^+$
- $S_i \overset{d}{=} F_{X_i}^{-1}(1-U)$ for $i \in I^-$

where $F_{S_i}(x)$ is the distribution function of S_i.

Super hedge portfolio

- Observe the inequality

$$\left(\sum_{i\in I+} w_i S_i - \sum_{i\in I-} |w_i| S_i - K\right)^+ \leq \sum_{i\in I+} w_i \left(S_i - \frac{\lambda_i K}{w_i}\right)^+ + \sum_{i\in I-} |w_i| \left(\frac{\lambda_i K}{|w_i|} - S_i\right)^+$$

  where $\lambda_i \geq 0$ and $\sum_{i\in I+} \lambda_i - \sum_{i\in I-} \lambda_i = 1$.

- Taking expectation on both sides of the inequality we have

$$\text{Spread option price} \leq \sum_{i\in I+} w_i C_{S_i}\left(\frac{\lambda_i K}{w_i}\right) + \sum_{i\in I-} |w_i| P_{S_i}\left(\frac{\lambda_i K}{|w_i|}\right)$$

  where $C_{S_i}(k)$ and $P_{S_i}(k)$ are the call and put prices of S_i struck at k respectively.

- The super hedge portfolio is therefore obtained by minimizing the right hand side over the constrained parameters $\lambda_1, \cdots, \lambda_n$.
- The portfolio consists of buying calls for the components with positive weight and puts for components with negative weights.

Optimal solution

- As in the basket case, the constrained minimization problem is solved by the method of Lagrange multiplier.
- Again the slopes $\Delta_j^{(i)}$ are ordered as a (strictly) decreasing sequence $\Delta_1, \cdots, \Delta_N$ with repetitions removed, where

$$\Delta_j^{(i)} = \frac{c_{j-1}^{(i)} - c_j^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^+$$

$$\Delta_j^{(i)} = \frac{p_j^{(i)} - p_{j-1}^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^-$$

- Correspond to each slope Δ_l, $\lambda_i(l) = \frac{w_i k^{(i)} j_i(l)}{K}$ is assigned to asset i, where

$$j_i(l) = \max\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^+$$

$$j_i(l) = \min\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^-$$

Optimal solution

- Start with $l = N$, each time let us decrease l by one until

$$\sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1.$$

Denote the critical l by l^*. If the condition $\sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1$ is skipped, linearly interpolate the λ_i's which changes while l decreases from l^* to $l^* - 1$. Denote the interpolation factor by θ^* and those indices by $I_{l^*}^+$ and $I_{l^*}^-$ for positive and negative weights respectively.

- Case I: $\sum_{i \in I^+} w_i k_i^{(i)} > K$ and $\sum_{i \in I^+} \lambda_i(l^*) - \sum_{i \in I^-} \lambda_i(l^*) = 1$

$$\text{UB} = \sum_{i \in I^+} C^{(i)} \left(\frac{w_i k_{j_i(l^*)}^{(i)}}{K} \right) + \sum_{i \in I^-} P^{(i)} \left(\frac{w_i k_{j_i(l^*)}^{(i)}}{K} \right)$$

Optimal solution

- Case II: $\sum_{i \in I^+} w_i k_i^{(i)} > K$ and $\sum_{i \in I^+} \lambda_i(l^*) - \sum_{i \in I^-} \lambda_i(l^*) > 1$

$$\begin{aligned}
\text{UB} \quad &= \sum_{i \in I^+ \setminus I^+_{l*}} w_i C^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right) + \sum_{i \in I^- \setminus I^-_{l*}} w_i P^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right) \\
&+ \sum_{i \in I^+_{l*}} w_i \left[\theta^* C^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right) + (1 - \theta^*) \theta^* C^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)-1}}{K} \right) \right] \\
&+ \sum_{i I^-_{l*}} w_i \left[\theta^* P^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right) + (1 - \theta^*) P^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)} + 1}{K} \right) \right]
\end{aligned}$$

- Case III: $\sum_{i \in I^+} w_i k_i^{(i)} \leq K$,

$$\text{UB} = \sum_{i \in I^+} w_i C^{(i)} (k_i^{(i)})$$

Simulation illustration

K	Hedging Price	MC Price	MC accuracy	S_1 strike	C	S_2 strike	P
2	10.03	10.12	0.07	1.46	0.16	59/59.5	3.43/3.17
2.5	9.77	9.71	0.07	1.46	0.16	58.5	3.17/2.92
3.5	9.29	9.29	0.07	1.48	0.15	58	2.68
4.5	8.83	8.83	0.06	1.48	0.15	57.5/58	2.68/2.46
13	5.60	5.64	0.05	1.65/1.60	0.09/0.1	54.5	1.35

S_1 and S_2 are distributed like two antimonotonic geometric Brownian motions (equivalently the instantaneous correlation ρ equals -1) with parameters $\sigma_1 = .355$, $\sigma_2 = .2$, $T = .5$, $r = 0$, $d_1 = d_2 = 0$. The Monte Carlo prices are computes using $n = 50,000$ paths. The spot prices are $S_1 = 1.48$, $S_2 = 59.33$, and the weights are $w_1 = 42$, $w_2 = 1$. The strikes that were actually trading are given by the NYMEX data for the December 2006 contract.

Empirical analysis

The results of monitoring the crack spread option, difference between heating and crude oil for the contract that expired December 2006 are shown in the following table. The table shows the true price in the third column and the lower and upper bounds in column 2 and 4. The comononotinicity and antimonotonicity gaps are shown next, as well as their relative counterparts.

Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
6-Oct	1.39	2.65	7.52	1.25	4.88	6.13	0.20	0.80
13-Oct	1.53	3.06	7.53	1.52	4.47	6.00	0.25	0.75
20-Oct	1.26	2.55	6.72	1.30	4.17	5.46	0.24	0.76
23-Oct	0.95	2.40	5.22	1.45	2.82	4.27	0.34	0.66
26-Oct	1.29	2.24	6.15	0.95	3.91	4.86	0.20	0.80
30-Oct	0.57	1.39	5.17	0.81	3.78	4.60	0.18	0.82
31-Oct	0.57	1.36	5.10	0.79	3.73	4.52	0.17	0.83
1-Nov	0.49	1.09	4.75	0.60	3.65	4.26	0.14	0.86
2-Nov	0.47	2.26	4.69	1.79	2.43	4.22	0.42	0.58
3-Nov	0.60	2.50	4.92	1.90	2.42	4.32	0.44	0.56
6-Nov	0.85	2.96	5.17	2.11	2.21	4.32	0.49	0.51
7-Nov	1.00	1.45	5.04	0.45	3.59	4.04	0.11	0.89
8-Nov	0.83	1.25	4.87	0.42	3.62	4.04	0.10	0.90
9-Nov	1.13	1.10	5.19	-0.03	4.09	4.05	-0.01	1.01

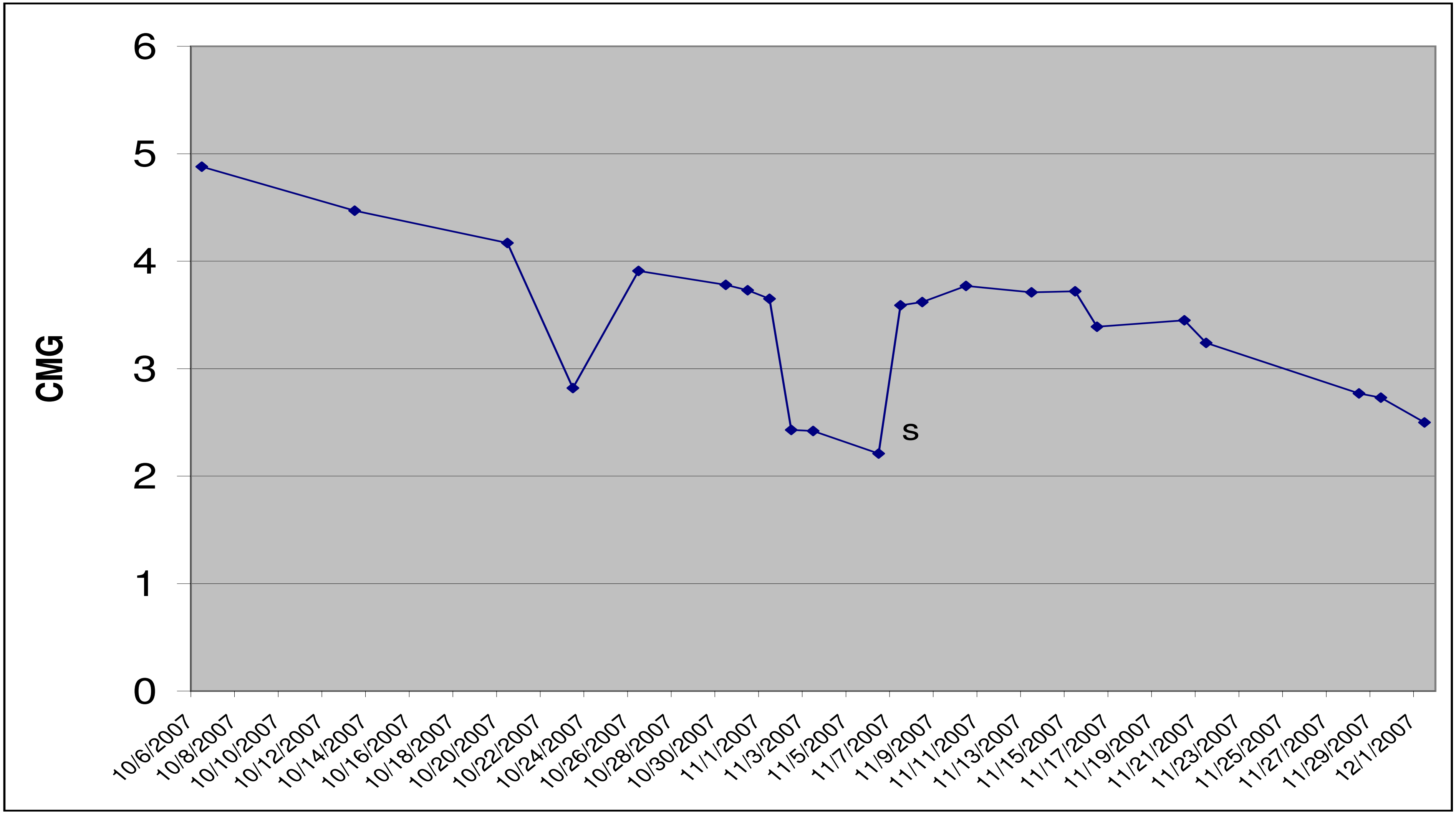
Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
10-Nov	0.87	1.10	4.87	0.23	3.77	4.00	0.06	0.94
13-Nov	0.60	0.65	4.36	0.05	3.71	3.76	0.01	0.99
14-Nov	0.93	0.80	4.69	-0.13	3.89	3.76	-0.04	1.04
15-Nov	1.05	1.15	4.87	0.10	3.72	3.83	0.03	0.97
16-Nov	1.21	1.53	4.92	0.32	3.39	3.71	0.09	0.91
20-Nov	1.36	1.37	4.82	0.01	3.45	3.46	0.00	1.00
21-Nov	2.13	2.23	5.47	0.10	3.24	3.35	0.03	0.97
28-Nov	1.35	1.51	4.28	0.16	2.77	2.93	0.05	0.95
29-Nov	2.10	2.10	4.83	0.00	2.73	2.73	0.00	1.00
1-Dec	1.70	1.75	4.25	0.05	2.50	2.55	0.02	0.98
4-Dec	1.30	1.20	3.69	-0.10	2.49	2.39	-0.04	1.04
5-Dec	1.09	0.82	3.35	-0.27	2.53	2.26	-0.12	1.12
6-Dec	1.03	0.97	3.14	-0.06	2.17	2.11	-0.03	1.03
7-Dec	0.72	0.56	2.64	-0.16	2.08	1.93	-0.08	1.08

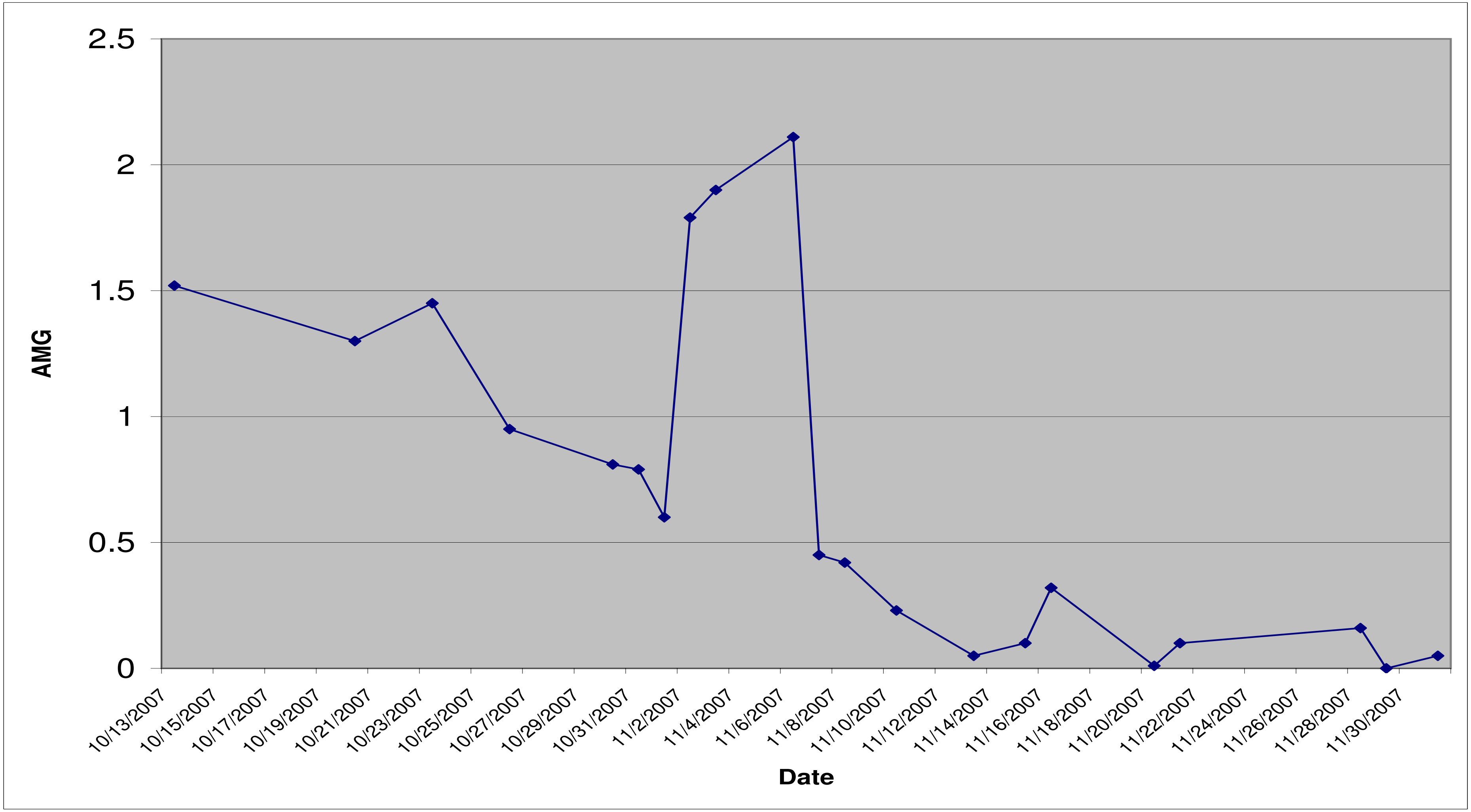
Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
8-Dec	0.64	0.38	2.36	-0.26	1.98	1.72	-0.15	1.15
11-Dec	0.50	0.15	1.84	-0.35	1.69	1.34	-0.26	1.26
12-Dec	0.53	0.14	1.74	-0.39	1.60	1.21	-0.33	1.33
13-Dec	0.62	0.16	1.56	-0.46	1.40	0.94	-0.49	1.49

Empirical analysis



Empirical analysis



Slogan: MIND THE GAP !

THANK YOU FOR YOUR PATIENCE !