Option Pricing and Hedging for Conditional Leptokurtic Returns

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Outline

1. Introduction

- European call option pricing formula:

$$C_0 = E_0^Q[e^{-rT} C_T] = E_0^Q[e^{-rT}(S_T - K)^+]$$

  - no-arbitrage price
  - risk-neutral probability measure Q

- Complete market model: unique no-arbitrage price

- Incomplete market model: no-arbitrage prices

- Black-Scholes model (1973): a complete market model

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is Brownian motion.

- European call option: $C_0 = S_0 N(d_1) + Ke^{-rT} N(d_2)$,

where $d_1(S_0) = \frac{\ln(S_0/K) + (r+0.5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- This option can be replicated by a self-financing trading strategy continuously with the initial capital C_0.

- In the Black-Scholes model,

$$\log \frac{S_T}{S_0} \sim N\Big((r - \frac{1}{2}\sigma^2)T, \sigma^2 T\Big)$$

- Financial data: heavy-tailed property
- Volatility smile: implied volatility is not a constant

- Conditional heteroscedastic models: incomplete models
  - ARCH model (Engle, 1982)
  - GARCH model (Bollerslev, 1986)

$$\begin{cases} R_t = \log \dfrac{S_t}{S_{t-1}} = r - \dfrac{1}{2}\sigma_t^2 + \lambda\sigma_t + \sigma_t\varepsilon_t, \quad \varepsilon_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \alpha_2\sigma_{t-1}^2 \end{cases}$$

- GARCH-normal option pricing: Duan (1995), Duan and Simonato (1998, 2001), Duan et al. (1999) and Ritchken and Trevor (1999)

- Financial data: conditional leptokurtic property

- Conditional leptokurtic models: Bollerslev (1987), Tong (1990), Nelson(1991), Shephard (1996), Mills (1999), Fan and Yao (2003)

$$\begin{cases} R_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \mid \mathcal{F}_{t-1} \sim D(0, \sigma_t^2) \\ \sigma_t^2 = f(\sigma_s, \varepsilon_s; -\infty < s \le t-1, \theta) \end{cases},$$

where the function $f(\cdot)$ and the set of parameters, θ, govern the volatility dynamics.

- Option pricing in conditional leptokurtic models: Elliott and Madan (1998), Duan (1999), Siu et al. (2004), Duan et al. (2005) and Christoffersen et al. (2006)

- Conditional leptokurtic model: no-arbitrage price is not unique
- Can we determine a no-arbitrage price from a view point of hedging in conditional leptokurtic model?
- Is this initial hedging capital related to a specific risk-neutral probability measure?

2 Hedging

- A hedge is a financial strategy used to reduce the risk of adverse price movements in an asset by buying or selling others.

- For example, when a trader writes a call option of a stock, she may set up the following hedging portfolio:

$$C_t = h_{0,t} B_t + h_{1,t} S_t, \ 0 \leq t < T,$$

where $C_t = e^{-r(T-t)} E_t^Q[(S_T - K)^+]$.

- How to determine the hedging positions $(h_{0,t}, h_{1,t})$?

2.1 Δ-hedging

- One of the most popular hedging strategy is the delta neutral hedging:

$$\begin{cases} h_{1,t} = \Delta_t \equiv \frac{\partial C_t}{\partial S_t} \\ h_{0,t} = (C_t - \Delta_t S_t)/B_t \end{cases}$$

where

$$\Delta_t = e^{-r(T-t)} E_t^Q \left(\frac{S_T}{S_t} I_{\{S_T \geq K\}} \right).$$

- In Black-Scholes model:

$$\Delta_t = \frac{\partial C_t}{\partial S_t} = e^{-r(T-t)} E_t^Q \left(\frac{S_T}{S_t} I_{\{S_T \geq K\}} \right) = N(d_1)$$

where $d_1(S_t) = \frac{\ln(S_t/K)+(r+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

- In GARCH models:
  1. Compute $E_t^Q(S_T I_{\{S_T \geq K\}})$ either by empirical martingale simulation (Duan and Simonato, 1998) or Dynamic Semiparametric Approach (Huang and Guo, 2009)
  2. Approximate $\frac{\partial C_t}{\partial S_t}$ by $\frac{C_t(S_t+h)-C_t(S_t)}{h}$, where h is a small constant.

2.2 Squared Risk Adjusted Hedging (η-hedging)

- Set up a hedging portfolio P_{t-1} at time $t-1$:

$$P_{t-1} = \eta^0_{t-1} B_{t-1} + \eta^1_{t-1} S_{t-1}$$

- The discounted value of this hedging portfolio at time $t-1$:

$$\tilde{P}_{t-1} = \frac{P_{t-1}}{B_{t-1}} = \eta^0_{t-1} + \eta^1_{t-1} \tilde{S}_{t-1},$$

where $\tilde{S}_{t-1} = S_{t-1}/B_{t-1}$.

- The discounted additional capital at time t:

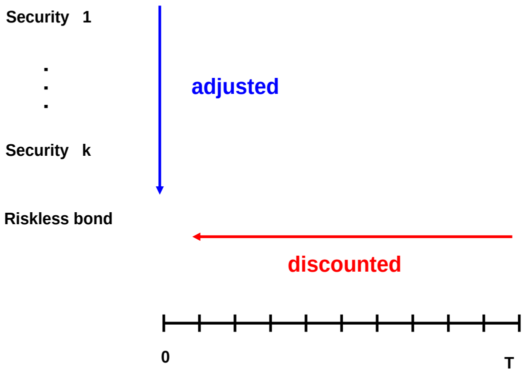
$$\tilde{\delta}_t = \tilde{V}_t - \tilde{P}_t = \tilde{V}_t - (\eta^0_{t-1} + \eta^1_{t-1} \tilde{S}_t),$$

where $\tilde{V}_t = V_t/B_t$.

- The discounted adjusted additional capital at time t:

$$\tilde{\delta}_t^* = \tilde{V}_t^* - (\eta_{t-1}^0 + \eta_{t-1}^1 \tilde{S}_t^*),$$

where $\tilde{S}_t^* = \tilde{S}_t e^{-u_t}$, $\tilde{V}_t^* = \tilde{V}_t(S_t e^{-u_t})$ and u_t is the risk premium.



- The optimal criterion for choosing $(\eta_{t-1}^{0}, \eta_{t-1}^{1})$:

$$(\hat{\eta}_{t-1}^{0}, \hat{\eta}_{t-1}^{1}) = \text{arg min}_{\eta_{t-1}^{0}, \eta_{t-1}^{1}} E_{t-1}(\tilde{\delta_t^*}^2).$$

- Minimum expected squared discounted risk adjusted hedging strategy (η-hedging)

- By solving $\frac{\partial}{\partial \eta_{t-1}^0} E_{t-1}(\tilde{\delta}_t^{*2}) = 0$ and $\frac{\partial}{\partial \eta_{t-1}^1} E_{t-1}(\tilde{\delta}_t^{*2}) = 0$, we have

$$\begin{cases} (\hat{\eta}_T^0, \hat{\eta}_T^1) = (\tilde{V}_T^*, 0) \\ \hat{\eta}_{t-1}^1 = \mathrm{Cov}_{t-1}(\tilde{V}_t^*, \tilde{S}_t^*)/\mathrm{Var}_{t-1}(\tilde{S}_t^*) \\ \hat{\eta}_{t-1}^0 = E_{t-1}(\tilde{V}_t^*) - \hat{\eta}_{t-1}^1 \tilde{S}_t^* = E_{t-1}\{\hat{\eta}_t^0 + (\hat{\eta}_t^1 - \hat{\eta}_{t-1}^1)\tilde{S}_t^*\} \end{cases},$$

for $t = 1, \cdots, T$.

- The initial discounted hedging capital of the η-hedging:

$$\tilde{P}_0 = \hat{\eta}_0^0 + \hat{\eta}_0^1 \tilde{S}_0$$

2.3 The consistency between η-hedging and the extended Girsanov change of measure

- The extended Girsanov change of measure: a discretized version of the Girsanov change of measure
- Elliott and Madan (1998):

$$\Lambda_t = \prod_{k=1}^{t} \frac{\phi_k(\tilde{S}_k/\tilde{S}_{k-1})e^{u_k}}{\phi_k(e^{-u_k}\tilde{S}_k/\tilde{S}_{k-1})}, \quad t = 0, 1, \cdots \tag{1}$$

where $u_k = \ln E_{k-1}(\tilde{S}_k/\tilde{S}_{k-1})$,
and $\phi_k(\cdot)$ is the conditional density of $e^{-u_k}\tilde{S}_k/\tilde{S}_{k-1}$ given $\mathcal{F}_{k-1}$.

- The positive process Λ_t is a P-martingale
- The martingale measure $Q(\cdot)$ is defined by $dQ = \Lambda_t dP$
- $\tilde{S}_t$ is a martingale under Q
- The law of $\frac{\tilde{S}_t}{\tilde{S}_{t-1}}$ given $\mathcal{F}_{t-1}$ under measure Q is equal to the law of $\frac{\tilde{S}_t}{\tilde{S}_{t-1}} e^{-u_t}$ given $\mathcal{F}_{t-1}$ under measure P.

- The consistency between η-hedging and the extended Girsanov change of measure:

  The time $t-1$ discounted hedging capital of η-hedging,

$$\tilde{P}_{t-1} = E_{t-1}(\tilde{V}_t^*) = E_{t-1}(\tilde{V}_t(S_t e^{-u_t})),$$

  is identical to the discounted risk-neutral price,

$$\tilde{V}_{t-1} = E_{t-1}^Q(\tilde{V}_t).$$

- In general, we show that

$$E_{t-1}[G(S_t e^{-u_t})] = E^Q_{t-1}[G(S_t)]$$

for any function $G(\cdot)$.

- Hence, we conclude that

$$\tilde{P}_{t-1} = E_{t-1}(\tilde{V}^*_t) = E_{t-1}(\tilde{V}_t(S_t e^{-u_t})) = E^Q_{t-1}(\tilde{V}_t) = \tilde{V}_{t-1}$$

- Moreover, the optimal criterion of the η-hedging can be rewritten as

$$\begin{aligned}(\hat{\eta}^0_{t-1}, \hat{\eta}^1_{t-1}) &= \arg\min_{\eta^0_{t-1}, \eta^1_{t-1}} E_{t-1}(\tilde{\delta}^{*2}_t) \\ &= \arg\min_{\eta^0_{t-1}, \eta^1_{t-1}} E^Q_{t-1}(\tilde{\delta}_t^2),\end{aligned}$$

- The optimal choice of $(\eta^0_{t-1}, \eta^1_{t-1})$ is also given by

$$\begin{cases} (\hat{\eta}^0_T, \hat{\eta}^1_T) = (\tilde{V}_T, 0) \\ \hat{\eta}^1_{t-1} = \mathrm{Cov}^Q_{t-1}(\tilde{V}_t, \tilde{S}_t)/\mathrm{Var}^Q_{t-1}(\tilde{S}_t) \\ \hat{\eta}^0_{t-1} = E^Q_{t-1}(\tilde{V}_t) - \hat{\eta}^1_{t-1}\tilde{S}_t \end{cases} .$$

- We also have

$$\tilde{P}_0 = \tilde{V}_0,$$

where $\tilde{V}_0$ is the initial option value derived by the risk-neutral measure Q.

2.4 An invariant property of the hedging period and the dynamic programming of η-hedging

- Suppose that investors decide to keep the holding units of the hedging portfolio constant for h units of time due to the impact of the transaction costs.

- Denote the discounted hedging capital with hedging period h at time $t-1$ by

$$\tilde{P}_{t-1,h} = \hat{\eta}^0_{t-1,h} + \hat{\eta}^1_{t-1,h}\tilde{S}_{t-1}, \tag{2}$$

where $\hat{\eta}^0_{t-1,h}$ and $\hat{\eta}^1_{t-1,h}$ are determined by the optimal criterion

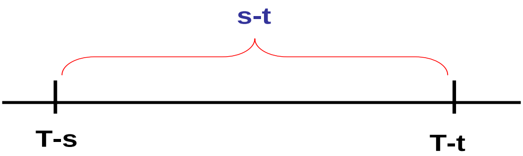
$$\min_{\eta^0_{t-1,h},\eta^1_{t-1,h}} E^Q_{t-1}[(\tilde{\delta}_{t-h+1}(S_{t-h+1}))^2], \tag{3}$$

- European call option with strike price K and maturity date T
- Hedging starts: $T - s$

  Hedging capital: $V_{T-s} = e^{-rT-s} E^{Q}_{T-s}\{(S_T - K)^+\}$

  Hedging target: V_{T-t}, $t < s$

  Hedging period : $s - t$



- Is $\tilde{P}_{t-1,h}$ a constant for $h = 1, 2, \cdots, T - t + 1$?

Proposition 1

Let $\tilde{P}_{t-1,h}$, defined in (2), be the discounted capital at time $t - 1$ for hedging the discounted derivative value function $\tilde{V}_{t+h-1}$ at time $t + h - 1$ derived by risk-neutral probability measure Q, where h is an integer and denotes the hedging period. If the holding units of riskless bonds and the underlying asset are determined by the optimal criterion (3), then $\tilde{P}_{t-1,h}$ is identical to $\tilde{V}_{t-1}$ for all $h = 1, 2, \cdots, T - t + 1$.

- The optimal holding units $(\hat{\eta}^0_{t-1,h}, \hat{\eta}^1_{t-1,h})$ can be represented as

$$\begin{cases} \hat{\eta}^1_{t-1,h} = \mathsf{Cov}^Q_{t-1}(\tilde{V}_{t+h-1}, \tilde{S}_{t+h-1})/\mathsf{Var}^Q_{t-1}(\tilde{S}_{t+h-1}) \\ \hat{\eta}^0_{t-1,h} = E^Q_{t-1}(\tilde{V}_{t+h-1}) - \hat{\eta}^1_{t-1,h}\tilde{S}_{t+h-1} \end{cases} . \quad (4)$$

- Equation (4) is handy in building η-hedging.

- The η-hedging is obtained as follows:

  1. For a given stock price S_{t-1} at time $t-1$ and a given hedging period h, generate n stock prices $\{S_{t+h-1,j}\}_{j=1}^{n}$, at time $t+h-1$ conditional on S_{t-1} from the risk-neutral model.
  2. For each $S_{t+h-1,j}$, derive the corresponding European call option prices, $V_{t+h-1}(S_{t+h-1,j})$, either by the DSA or EMS method. If $h = T - t + 1$, then $V_T(S_{T,j}) = (S_{T,j} - K)^+$.
  3. Regress $\tilde{V}_{t+h-1}(S_{t+h-1,j})$ on $\tilde{S}_{t+h-1,j}$, $j = 1, \cdots, n$. Then $(\hat{\eta}^0_{t-1}, \hat{\eta}^1_{t-1})$ are the corresponding regression coefficients.

2.5 η-hedging and Δ-hedging under complete and incomplete markets

- Complete market: every contingent claim is marketable and the risk neutral probability measure is unique.
- There exists a self-financing trading strategy and the holding units of the stocks and bonds in the replicating portfolio are uniquely determined.
- We expect the η-hedging will coincide with the Δ-hedging under the complete market models.

- The binomial tree model:
  - Suppose that $S_{t-1} = s$ and goes up to $s_u = us$ or down to $s_d = ds$ at time t with risk neutral probability

$$P(S_t = s_u) = \frac{e^r - d}{u - d} = 1 - P(S_t = s_d).$$

  - By straightforward computation, we have

$$\eta_{t-1}^1 = \frac{V_t(s_u) - V_t(s_d)}{s_u - s_d},$$

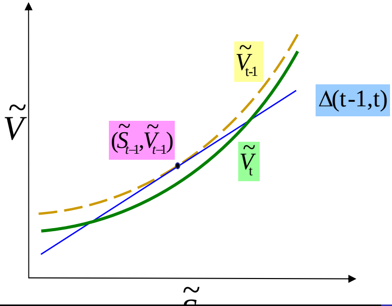
which is the same as the Δ-hedging in a binomial tree model.

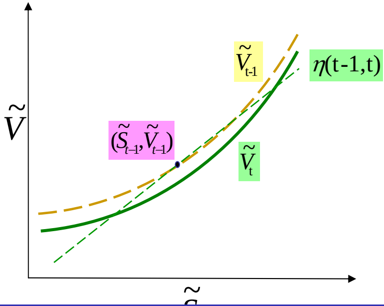
- The Black-Scholes model: $dS_t = rS_t dt + \sigma S_t dW_t$
  - The Δ-hedging is $N(d_1(S_t))$ at time t.
  - We show that $\eta_t^1 \to N(d_1)$ as $dt \to 0$, where dt denotes the length of the time period $[t, t+dt]$.

- The difference of the expected discounted squared risks between Δ-hedging and η-hedging with hedging period h under the risk-neutral probability measure Q.

- Hedging capital: $\tilde{V}_{t-1}$

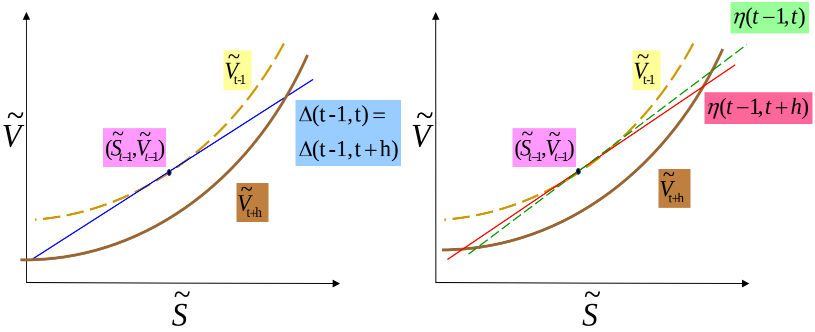
  Hedging target: $\tilde{V}_t$





- Hedging capital: $\tilde{V}_{t-1}$

  Hedging target: $\tilde{V}_{t+h}$



- Let $a + b\tilde{S}_t$ be a discounted hedging portfolio at time t and the initial discounted hedging capital equals $\tilde{V}_t$.

- Denote

$$H_t(b) = E_t^Q\{[\tilde{V}_{t+h} - (a + b\tilde{S}_{t+h})]^2\},$$

where the coefficients a and b satisfy $a + b\tilde{S}_t = \tilde{V}_t$.

- By straightforward computation, we have

$$\begin{aligned} H_t(b) &= \mathrm{Var}_t^Q(\tilde{V}_{t+h}) - 2b\mathrm{Cov}_t^Q(\tilde{V}_{t+h}, \tilde{S}_{t+h}) + b^2\mathrm{Var}_t^Q(\tilde{S}_{t+h}) \\ &= \mathrm{Var}_t^Q(\tilde{S}_{t+h})(b - \hat{\eta}_{t,h}^1)^2 + H_t(\hat{\eta}_{t,h}^1) \geq H_t(\hat{\eta}_{t,h}^1), \end{aligned}$$

where $\hat{\eta}_{t,h}^1$ is computed by (4) and

$$H_t(\hat{\eta}_{t,h}^1) = \frac{\mathrm{Var}_t^Q(\tilde{V}_{t+h})\mathrm{Var}_t^Q(\tilde{S}_{t+h}) - [\mathrm{Cov}_t^Q(\tilde{V}_{t+h}, \tilde{S}_{t+h})]^2}{\mathrm{Var}_t^Q(\tilde{S}_{t+h})}.$$

Corollary 1

Suppose the underlying assets follows the Black-Scholes model. Define $H_t(\Delta_t)$ and $H_t(\hat{\eta}^1_{t,h})$ as above. Then we have

$$H_t(\Delta_t) - H_t(\hat{\eta}^1_{t,h}) = O\Big(\frac{h^3}{T-t-h} \exp\Big\{ - \frac{(\ln \frac{S_t}{K})^2}{T-t-h} \Big\}\Big),$$

where T is the expiration date and h is the hedging time length.

- $H_t(\Delta_t) - H_t(\hat{\eta}^1_{t,h})$ approaches to zero for both OTM and ITM cases for any fixed hedging time length h.
- $H_t(\Delta_t) - H_t(\hat{\eta}^1_{t,h})$ approaches to zero as $h \to 0$.

3. Risk-neutral model and option valuation

- Assume the underlying asset log return process $\{R_t\}$ under the physical measure P satisfies the following:

$$\begin{cases} R_t = \mu_t - \gamma_t + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim D(0,1) \\ \sigma_t^2 = f(\sigma_s, \varepsilon_s; -\infty < s \leq t-1, \theta) \end{cases}, \tag{5}$$

where σ_t^2 is the conditional variance of the log return in period t and the mean correction factor γ_t is defined as $\gamma_t = \ln\{E_{t-1}(e^{\sigma_t \varepsilon_t})\}$, which serves to ensure $E_{t-1}(S_t) = S_{t-1}e^{\mu_t}$, where $E_{t-1}(\cdot)$ denotes the conditional expectation given $\mathcal{F}_{t-1}$ under the physical measure P, and the function $f(\cdot)$ and the set of parameters, θ, govern the volatility dynamic.

Theorem 1

Assume the dynamic of the logarithmic return process $\{R_t\}$ under the physical measure P satisfies Model (5), then the risk-neutral model derived by the extended Girsanov principle with change of measure density (1) is

$$\begin{cases} R_t = r_t - \gamma_t + \sigma_t \xi_t, \quad \xi_t \sim D(0,1) \\ \sigma_t^2 = f(\sigma_s, \xi_s + \frac{r_s - \mu_s}{\sigma_s}; -\infty < s \leq t-1, \theta) \end{cases}, \tag{6}$$

and the law of $\xi_t = \varepsilon_t - \frac{r_t - \mu_t}{\sigma_s}$ under the risk-neutral measure Q is the same as that of ε_t under the physical measure P.

- In particular, for the GARCH-normal dynamic model considered by Duan (1995)

$$\begin{cases} R_t = r - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t + \sigma_t\varepsilon_t, \quad \varepsilon_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \alpha_2\sigma_{t-1}^2 \end{cases}, \tag{7}$$

the risk-neutralized GARCH-normal model,

$$\begin{cases} R_t = r - \frac{1}{2}\sigma_t^2 + \sigma_t\xi_t, \quad \xi_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - \lambda)^2 + \beta_1\sigma_{t-1}^2 \end{cases}, \tag{8}$$

is the same when applying the extended Girsanov martingale probability and the local risk-neutral valuation relationship introduced by Duan (1995).

- In the case of deriving the risk-neutralized GARCH models without conditional cumulant-generating function, such as the t-distributions, we utilize the following simple return model instead,

$$\begin{cases} \tilde{R}_t \equiv \frac{S_t - S_{t-1}}{S_{t-1}} = \mu_t + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim D(0,1). \\ \sigma_t^2 = f(\sigma_s, \varepsilon_s; -\infty < s \leq t-1, \theta) \end{cases} . \qquad (9)$$

- To ensure the nonnegativeness of the asset prices in a simple return model, the lower bound of the simple return is -1.

- In the case of leptokurtic t innovations with degrees of freedom greater than 2, the probability of having negative asset prices in Model (9) is

$$P(S_t < 0 \mid S_{t-1} > 0) = P(1 + \mu_t + \sigma_t \varepsilon_t < 0 \mid S_{t-1} > 0)$$
$$= P(T_\nu < -\sqrt{\nu/\nu - 2}(\mu_t + 1)/\sigma_t \mid S_{t-1} > 0),$$

  where $T_\nu = \sqrt{\nu/\nu - 2}\varepsilon_t$ has a t distribution with ν degrees of freedom, which is in general very small and can be neglected in practice.

- For example, if the degrees of freedom is 5, the daily volatility $\sigma_t = 0.5/\sqrt{250} = 0.0316$, and $\mu_t > 0$, then the probability $P(T_\nu < -40.85)$ is almost 0.

Theorem 2

Assume the dynamic of the simple return process $\{\tilde{R}_t\}$ under the physical measure P satisfies Model (9), then the risk-neutral model derived by the extended Girsanov principle with change of measure density (1) is

$$\begin{cases} \tilde{R}_t = \tilde{r}_t + \frac{1+\tilde{r}_t}{1+\mu_t}\sigma_t\xi_t, \quad \xi_t \sim D(0,1) \\ \sigma_t^2 = f(\sigma_s, \frac{1+\tilde{r}_s}{1+\mu_s}\xi_s + \frac{\tilde{r}_s-\mu_s}{\sigma_s}; -\infty < s \leq t-1, \theta) \end{cases}, \tag{10}$$

where $\tilde{r}_t$ is the riskless simple interest rate in the time period $[t-1, t)$ and is $\mathcal{F}_{t-1}$-measurable. The law of $\xi_t = \frac{1+\mu_t}{1+\tilde{r}_t}(\varepsilon_t - \frac{\tilde{r}_t-\mu_t}{\sigma_s})$ under the risk-neutral measure Q is the same as that of ε_t under the physical measure P.

4. Simulation

- The average squared hedging costs of Δ-hedging and η-hedging at time $T-s$:

$$\begin{cases} A^{\Delta}_{T-s} & = & \sum_{j=1}^{n}[V_T(S_{T,j}) - (\Delta^0_{T-s}e^{rs} + \Delta^1_{T-s}S_{T,j})]^2 \\ A^{\eta}_{T-s} & = & \sum_{j=1}^{n}[V_T(S_{T,j}) - (\eta^0_{T-s}e^{rs} + \eta^1_{T-s}S_{T,j})]^2 \end{cases}$$

- The relative error of A^{Δ}_{T-s} and A^{η}_{T-s}:

$$D_{T-s} = (A^{\Delta}_{T-s} - A^{\eta}_{T-s})/A^{\eta}_{T-s}.$$

Table1: The relative errors of the average squared hedging costs of Δ-hedging and η-hedging for European call options in the GARCH(1,1) log-return models

	GARCH-normal			GARCH-dexp		
	$K = 35$	$K = 40$	$K = 45$	$K = 35$	$K = 40$	$K = 45$
D_{T-1}	0.00%	0.02%	0.00%	0.00%	0.18%	0.02%
D_{T-5}	0.41%	0.08%	0.85%	1.83%	0.22%	3.55%
D_{T-10}	1.08%	0.14%	2.14%	3.38%	0.41%	7.30%
D_{T-30}	1.74%	0.38%	4.32%	4.69%	0.89%	11.17%

Delta values of GARCH-t and GARCH-normal models

- Assume the underlying asset following an NGARCH-t model:

$$\begin{cases} \tilde{R}_t = \frac{0.05}{365} + 0.2\sigma_t + \sigma_t\varepsilon_t, \quad \varepsilon_t \sim \sqrt{3/5}\ t(5) \\ \sigma_t^2 = 0.0001 + 0.1\sigma_{t-1}^2(\varepsilon_{t-1} - 0.3)^2 + 0.8\sigma_{t-1}^2, \end{cases}$$

where $\frac{0.05}{365}$ is the daily riskless interest rate, the risk premium is given by 0.2 and the leverage effect parameter is 0.3.

- By Theorem 2, the risk-neutral model is as follows:

$$\begin{cases} \tilde{R}_t = \frac{0.05}{365} + \frac{1+0.05/365}{1+0.05/365+0.2\sigma_t}\sigma_t\xi_t, \quad \xi_t \sim \sqrt{3/5}\, t(5) \\ \sigma_t^2 = 0.0001 + 0.1\sigma_{t-1}^2\left(\frac{1+0.05/365}{1+0.05/365+0.2\sigma_{t-1}}\xi_{t-1} - 0.5\right)^2 \\ \quad\quad +0.8\sigma_{t-1}^2. \end{cases} \tag{11}$$

- The market prices of European call option with maturity dates $T = 10, 30, 60$ and strike prices $K = 45, 47.5, 50, 52.5, 55$ are obtained by the EMS method with Model (11).

- Fit a risk-neutral NGARCH-normal model by minimizing the sum of squared error between the market prices and NGARCH-normal model prices:

$$\left\{ \begin{array}{l} R_t = \frac{0.05}{365} - \frac{1}{2}\sigma_t^2 + \sigma_t^2\xi_t, \quad \xi_t \sim N(0,1) \\ \sigma_t^2 = 0.00002 + 0.1167\sigma_{t-1}^2(\xi_{t-1} - 0.7898)^2 + 0.6450\sigma_{t-1}^2. \end{array} \right. \tag{12}$$

Table 2: European call option and delta values of Model (11) and (12)

	K	32	36	40	44	48
T = 10 days	C	8.0010	4.0129	0.5862	0.0176	0.0020
		(8.0010)	(4.0135)	(0.5821)	(0.0200)	(0.0027)
	Δ_0^H	0.9996	0.9912	0.5087	0.0139	0.0011
		(0.9996)	(0.9908)	(0.5073)	(0.0148)	(0.0013)
T = 30 days	C	8.0108	4.0903	1.0240	0.1218	0.0218
		(8.0106)	(4.0894)	(1.0216)	(0.1246)	(0.0245)
	Δ_0^H	0.9964	0.9512	0.5139	0.0722	0.0094
		(0.9966)	(0.9517)	(0.5129)	(0.0719)	(0.0098)
T = 60 days	C	8.0373	4.2601	1.4687	0.3381	0.0787
		(8.0358)	(4.2576)	(1.4697)	(0.3403)	(0.0820)
	Δ_0^H	0.9876	0.8912	0.5195	0.1515	0.0327
		(0.9879)	(0.8907)	(0.5182)	(0.1514)	(0.0331)

The values in the parentheses are of Model (12).

- Asian call option:

$$\max\left(\frac{1}{T}\sum_{t=1}^{T} S_t - K, 0\right)$$

- Lookback call option:

$$\max(S_T - m_T, 0),$$

where $m_T = \min_{0 \le t \le T} S_t$.

- Barrier (up-and-out) call option:

$$\max(S_T - K, 0) I_{\{S_t \le B, \forall\ 0 \le t \le T\}}$$

Table 3: Asian call option and delta values of Model (11) and (12)

	K	32	36	40	44	48
$T=10$ days	C	8.0001	4.0015	0.3669	0.0023	0.0002
		(8.0000)	(4.0010)	(0.3673)	(0.0020)	(0.0001)
	Δ_0^H	1.0000	0.9988	0.5295	0.0022	0.0001
		(1.0000)	(0.9990)	(0.5284)	(0.0023)	(0.0001)
$T=30$ days	C	8.0010	4.0125	0.6065	0.0191	0.0026
		(8.0009)	(4.0124)	(0.6060)	(0.0206)	(0.0030)
	Δ_0^H	0.9996	0.9914	0.5217	0.0157	0.0012
		(0.9996)	(0.9913)	(0.5208)	(0.0163)	(0.0014)
$T=60$ days	C	8.0037	4.0430	0.8551	0.0649	0.0098
		(8.0039)	(4.0433)	(0.8557)	(0.0655)	(0.0101)
	Δ_0^H	0.9987	0.9729	0.5196	0.0465	0.0045
		(0.9987)	(0.9728)	(0.5195)	(0.0467)	(0.0046)

The values in the parentheses are of Model (12).

Table 4: Up-and-out and lookback call option and delta values of Model (11) and (12)

		$B = 44$		$B = 48$		$B = 52$		Lookback
	K	36	40	36	40	36	40	option
T = 10 days	C	3.8623	0.5044	3.9965	0.5743	4.0086	0.5825	0.8503
		(3.9180)	(0.5294)	(4.0135)	(0.5821)	(4.0136)	(0.5821)	(0.9422)
	Δ_0^H	0.8450	0.4461	0.9811	0.5167	0.9895	0.5224	0.0213
		(0.8184)	(0.4327)	(0.9911)	(0.5229)	(0.9912)	(0.5230)	(0.0236)
T = 30 days	C	3.2041	0.5464	3.9344	0.9150	4.0441	0.9874	1.6952
		(3.0375)	(0.4959)	(4.0145)	(0.9699)	(4.0874)	(1.0198)	(1.8645)
	Δ_0^H	0.3517	0.2043	0.8661	0.4629	0.9328	0.5074	0.0424
		(0.2007)	(0.1458)	(0.8684)	(0.4642)	(0.9504)	(0.5191)	(0.0466)
T = 60 days	C	2.2903	0.3958	3.6990	1.0813	4.0820	1.3315	2.5352
		(2.0615)	(0.3554)	(3.6928)	(1.0978)	(4.1736)	(1.4065)	(2.7458)
	Δ_0^H	-0.0010	0.0421	0.6194	0.3380	0.8171	0.4670	0.0634
		(-0.0439)	(0.0300)	(0.5549)	(0.3013)	(0.8238)	(0.4724)	(0.0686)

The values in the parentheses are of Model (12).

4. Conclusion

- A dynamic programming of η-hedging is proposed.
- The initial hedging capital of the η-hedging coincides to the option price obtained by the extended Girsanov change of measure.
- The η-hedging is identical to the commonly used Δ-hedging in the binomial tree model and Black-Scholes model.

- The η-hedging is more adapted to the hedging period than the Δ-hedging.
- The risk-neutral GARCH model is derived by the extended Girsanov principle.
- There is significant model risk when applying a conditional less-leptokurtic model to pricing some exotic options for the conditional leptokurtic returns.

- The η-hedging is more adapted to the hedging period than the Δ-hedging.
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References

- Black, F and Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Political Economy*, **81**, 637-654.
- Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Economet.*, **31**, 307-327.
- Bollerslev, T. (1987). A conditional heteroskedastic time series model for speculative prices and rates of return. *Review of Economics and Statistics*, **69**, 524-547.
- Christoffersen, P., Heston, S. and Jacobs, K. (2006). Option valuation with conditional skewness. *J. Econometrics*, **131**, 253-284.

- Duan, J. C. (1995). The GARCH option pricing model. *Math. Finance*, **5**, 13-32.
- Duan, J. C. (1999). Conditionally fat-tailed distributions and the volatility smile in options. Manuscript, University of Toronto.
- Duan, J. C., Gauthier, G. and Simonato, J. G. (1999). An analytical approximation for the GARCH option pricing model. *J. Computational Finance*, **2**, 75-116.
- Duan, J. C., Ritchken, p. and Sun, Z. (2005). Jump starting GARCH: pricing and hedging options with jumps in returns and volatilities. Working paper, Rotman school, University of Toronto.

- Duan, J. C. and Simonato, J. G. (1998). Empirical martingale simulation for asset prices. *Management Science*, **44**, 1218-1233.
- Duan, J. C. and Simonato, J. G. (2001). American option pricing under GARCH by a Markov chain approximation. *J. Economic Dynamics and Control*, **25**, 1689-1718.
- Elliott, R. J. and Madan, D. B. (1998). A discrete time equivalent martingale measure. *Math. Finance*, **8**, 127-152.
- Engle, R. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*, **50**, 987-1008.

- Fan, J. and Yao, Q. (2003). *Nonlinear Time Series: Nonparametric and Parametric Models.* Springer-Verlag: New York.
- Huang, S. F. and Guo, M. H. (2009). Financial derivative valuation - a dynamic semiparametric approach. *Statistica Sinica*, to appear.
- Mills, T. C. (1999). *The Econometric Modeling of Financial Time Series.* Cambridge University Press.
- Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns. *Econometrica*, **59**, 347-370.

- Ritchken, P. and Trevor, R. (1999). Pricing option under generalized GARCH and stochastic volatility processes. *J. Finance*, **8**, 377-402.
- Shephard, N. (1996). *Stochastic Aspects of ARCH and Stochastic Volatility,* in Cox, D. R., Hinkley, D. V., and Barndorff-Nielsen, O. E. (eds.), *Time Series Models: In Econometrics, Finance and Other Fields.* Chapman and Hall: New York.
- Siu, T. K., Tong, H. and Yang, H. (2004). On pricing derivatives under GARCH models: a dynamic Gerber-Shiu approach. *North American Actuarial Journal*, **8**, 17-31.
- Tong, H. (1990). *Nonlinear Time Series Analysis: A Dynamics Approach.* Oxford University Press.

Thanks For Your Attentions!