Chaper 4: Continuous-time interest rate models

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Agenda

- 4.1 One-factor models for the risk-free rate
- 4.2 The martingale approach

One-factor models for the risk-free rate

- For the one-factor models we will assume that

$$dr(t) = a(r(t))\,dt + b(r(t))\,dW(t)$$

  so that the process $r(t)$ is Markov and time homogeneous.

- Three desirable but not essential basic characteristics:
  - Interest rates should be positive.
  - $r(t)$ should be autoregressive.
  - Simple formulae for bond prices and some derivative prices.

The martingale approach

Suppose that

$$dr(t) = a(t)\,dt + b(t)\,dW(t) \quad (4.1)$$

$$dP(t,T) = P(t,T)[m(t,T)\,dt + S(t,T)\,dW(t)] \quad (4.2)$$

$r(t)$: risk-free interest rate
$P(t,T)$: price of a zero-coupon bond with maturity T

Note risk premium of $P(t,T) = m(t,T) - r(t)$

Note market price of risk $:= \gamma(t) = \dfrac{m(t,T) - r(t)}{S(t,T)}$

Note risk-free cash account

$$dB(t) = r(t)B(t)dt \text{ (i.e. } B(t) = B(0)e^{\int_0^t r(u)\,du})$$

The martingale approach

Given (4.1) and (4.2) and $0 < t < S < T$.

Consider an interest rate derivative contract which pays X_S at time S. What is the no-arbitrage price, $V(t)$, at time t?

Theorem 4.1

There exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$V(t) = E_{\mathbb{Q}}[e^{-\int_t^S r(u)\,du} X_S \mid \mathcal{F}_t]$$

where $dr(t) = (a(t) - \gamma(t)b(t))\,dt + b(t)\,d\tilde{W}(t)$ and $\tilde{W}(t)$ is

a standard Brownian motion umder $\mathbb{Q}$

The martingale approach

Proof of Theorem 4.1

$$Z(t,T) := \frac{P(t,T)}{B(t)} = P(t,T)e^{-\int_0^t r(u)\,du}$$

We now break the proof up into five steps.

Step 1

Claim: $\exists \mathbb{Q} \sim \mathbb{P}$ s.t. $Z(t,T)$ is a martingale.

Note

$$d(B(t)^{-1}) = -\frac{1}{B(t)^2}dB(t) + \frac{1}{2}\frac{2}{B(t)^3}d\langle B\rangle(t) = -\frac{r(t)}{B(t)}dt$$

The martingale approach

$\tilde{W}(t) := W(t) + \int_0^t \gamma(u)\, du$

Assume that $\gamma(s)$ satisfies the *Novikov* condition

$$\mathbb{E}_{\mathbb{P}}[e^{\frac{1}{2}\int_t^S \gamma(u)^2\, du}] < \infty,$$

By *Girsanov* theorem, $\exists \mathbb{Q} \sim \mathbb{P}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_t^S \gamma(u)\, du - \frac{1}{2}\int_t^S \gamma(u)^2\, du}$$

and under which $\tilde{W}(t)$ is a standard Brownian motion.

The martingale approach

$$dZ(t,T) = B(t)^{-1}dP(t,T) + P(t,T)d(B(t)^{-1}) + d\langle B^{-1}, P\rangle(t)$$

$$= \frac{P(t,T)}{B(t)}[m(t,T)dt + S(t,T)dW(t)] - P(t,T)\frac{r(t)}{B(t)}dt + 0$$

$$= Z(t,T)[(m(t,T) - r(t))dt + S(t,T)dW(t)]$$

$$= Z(t,T)[m(t,T) - r(t) - \gamma(t)S(t,T)dt + S(t,T)(dW(t) + \gamma(t)dt)]$$

$$= Z(t,T)S(t,T)d\tilde{W}(t)$$

$$\begin{aligned} d\log Z(t,T) &= \frac{1}{Z(t,T)}dZ(t,T) - \frac{1}{2}\frac{1}{Z(t,T)^2}Z(t,T)^2 S(t,T)^2\,dt \\ &= S(t,T)d\tilde{W}(t) - \frac{1}{2}S(t,T)^2\,dt \end{aligned}$$

The martingale approach

$$\therefore Z(S,T) = Z(t,T)e^{\int_t^S S(u,T)d\tilde{W}(u) - \frac{1}{2}\int_t^S S(u,T)^2 du}$$

which is a martingale if $\mathbb{E}_{\mathbb{Q}}[e^{\frac{1}{2}\int_t^S S(u,T)^2 du}] < \infty$

Note(Novikov condition)

If $\gamma(t)$ satisfies

$$\mathbb{E}_{\mathbb{P}}[e^{\frac{1}{2}\int_0^T \gamma(u)^2\,du}] < \infty$$

then

$$Z(t) := e^{-\int_0^t \gamma(u)\,dW(u) - \frac{1}{2}\int_0^t \gamma(u)^2\,du}$$

is a martingale under $\mathbb{P}$ for $0 \le t \le T$.

The martingale approach

Step 2

Given $t < t' < S$

Claim: $D(t) := \mathbb{E}_{\mathbb{Q}}[B(S)^{-1} X_S \,|\, \mathcal{F}_t]$ is a $\mathbb{Q}$-martingale

$\mathbb{E}_{\mathbb{Q}}[D(t') \,|\, \mathcal{F}_t]$

$= \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[B(S)^{-1} X_S \,|\, \mathcal{F}_{t'}] \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[B(S)^{-1} X_S \,|\, \mathcal{F}_t] \,|\, \mathcal{F}_{t'}]$

$= \mathbb{E}_{\mathbb{Q}}[B(S)^{-1} X_S \,|\, \mathcal{F}_t] = D(t)$

The martingale approach

Step 3

Claim: there exists a previsible process $\phi(t)$ s.t.

$$D(t) = D(0) + \int_0^t \phi(u) dZ(u, T)$$

By martingale representation theorem, $dD(t) = d'(t)\, d\tilde{W}(t)$.

Recall that $dZ(t, T) = Z(t, T)S(t, T)d\tilde{W}(t)$.

$$\begin{aligned}\therefore dD(t) &= \frac{d'(t)}{Z(t, T)S(t, T)} Z(t, T)S(t, T)\, d\tilde{W}(t) \\ &= \frac{d'(t)}{Z(t, T)S(t, T))}\, dZ(t, T) := \phi(t)\, dZ(t, T)\end{aligned}$$

The martingale approach

Step 4

$\psi(t) := D(t) - \phi(t)Z(t,T)$.Consider a portfolio as follows:

$\phi(t)$ units of $P(t,T)$ and $\psi(t)$ units of $B(t)$.

Claim: the portfolio above is self-financing.

Proof

The value of this portfolio at time t is

$$V(t) = \phi(t)P(t,T) + \psi(t)B(t) = B(t)[\phi(t)Z(t,T) + \psi(t)]$$

$$= B(t)D(t)$$

The martingale approach

$$dV(t) = d[B(t)D(t)] = B(t)dD(t) + D(t)dB(t) + dB(t)dD(t)$$

$$= B(t)\phi(t)dZ(t,T) + D(t)r(t)B(t)dt$$

$$= \phi(t)B(t)S(t,T)Z(t,T)d\tilde{W}(t) + (\phi(t)Z(t,T) + \psi(t))r(t)B(t)dt$$

$$= \phi(t)P(t,T)[r(t)dt + S(t,T)d\tilde{W}(t)] + \psi(t)r(t)B(t)dt$$

$$= \phi(t)dP(t,T) + \psi(t)dB(t)$$

Note $dP(t,T) = P(t,T)[m(t,T)\,dt + S(t,T)\,dW(t)]$

$$= P(t,T)[m(t,T)\,dt - S(t,T)\gamma(t)\,dt + S(t,T)\gamma(t)\,dt + S(t,T)\,dW(t)]$$

$$= P(t,T)[r(t)dt + S(t,T)d\tilde{W}(t)]$$

The martingale approach

$$V(t) = \phi(t)P(t,T) + \psi(t)B(t)$$

$$V(t+dt)' = \phi(t)[P(t,T) + dP(t,T)] + \psi(t)[B(t) + dB(t)]$$

$$= \phi(t)P(t,T) + \psi(t)B(t) + [\phi(t)dP(t,T) + \psi(t)dB(t)]$$

$$= V(t) + dV(t) = V(t+dt)$$

The instantaneous change in the portfolio value from t to $t+dt$ is equal to the instantaneous investment gain over the same period, so the portfolio process is self-financing.

The martingale approach

Step 5

$$V(t) = B(t)D(t) = \mathbb{E}_{\mathbb{Q}}[\frac{B(t)}{B(S)} X_S \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^S r(u)\,du} \,|\, \mathcal{F}_t]$$

$$\therefore V(S) = B(S)\mathbb{E}_{\mathbb{Q}}[B(S)^{-1} X_S \,|\, \mathcal{F}_S] = X_S$$

This implies not only that the portfolio process is self-financing but also that it replicates the derivative payoff. It follows that, for $t < S$, $V(t)$ is the unique no-arbitrage price at time t for X_S payable at S.

The martingale approach

Corollary 4.2

For all S s.t. $0 < S < T$,

$$P(t,S) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^S r(u)\,du} \,|\, \mathcal{F}_t]$$

<u>Remark</u>

$$dV(t) = \phi(t)dP(t,T) + \psi(t)dB(t)$$

$$= \phi(t)P(t,T)[r(t)dt + S(t,T)d\tilde{W}(t)] + \psi(t)B(t)r(t)dt$$

$$= [\phi(t)P(t,T) + \psi(t)B(t)]r(t)dt + \phi(t)P(t,T)S(t,T)d\tilde{W}(t)$$

$$:= V(t)[r(t)dt + \sigma_V(t)d\tilde{W}(t)]$$

The martingale approach

Under the risk-neutral measure $\mathbb{Q}$, the prices of all tradable assets have the risk-free rate of interest as the expected growth rate.

Now consider the price dynamics under the real-world measure $\mathbb{P}$

$$dV(t) = V(t)[r(t)dt + \sigma_V(t)(dW(t) + \gamma(t)dt)]$$

$$= V(t)[(r(t) + \gamma(t)\sigma_V(t))dt + \sigma_V(t)dW(t)]$$

In a one-factor model, the risk-premiums on different assets can differ only through the volatility in the tradable asset.