

A Hybrid Importance Sampling Algorithm for VaR

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Abstract. Value at Risk (VaR) provides a number that measures the risk of a financial portfolio under significant loss. The Monte Carlo simulation is a powerful tool for estimating VaR. However, the Monte Carlo simulation is inefficient since the event of significant loss is usually rare. Glasserman *et al.* suggest that the performance can be improved by importance sampling [4, 5]. To be brief, they sample the stock prices from a new probability measure where the event of significant loss is more likely to happen than in the original one. However, their technique might perform poorly for some complex portfolios like shorting straddle options. Shorting straddle options suffers significant loss when there is a significant increase or decrease in the underlying stock price. Thus tilting the probability measure of the stock price to make one huge-loss event, says a significant decrease in the stock price, more likely to happen will make the other event (a significant increase in the stock price) much more rare. The hybrid importance sampling algorithm suggested in this paper can efficiently estimate the VaR for complex portfolios. Take shorting straddle options for example, our algorithm can be decomposed into two sub simulations. Each sub simulation focuses on one huge-loss event like the significant decrease (or increase) in the stock price and tilts the probability measure to asymptotically minimize the variance for estimating the probability of that event. We proceed to minimize the variance for estimating the probability of significant loss given a limit on total computational time. The number of stock price samples serves as a proxy of computational time and the allotment of samples to each sub simulation is then properly determined by Lagrange’s multiplier. Our paper will demonstrate how the hybrid importance sampling method is applied to estimate the VaR of shorting straddle options by assuming that the stock price follows the Merton’s jump diffusion process [7]. Numerical experiments are given to verify the superiority of our method.

Keywords: hybrid importance sampling, VaR, straddle options, jump diffusion process

1 Introduction

Value at Risk (VaR) is an important tool for quantifying and managing portfolio risk. It provides a way of measuring the total risk to which the financial institution is exposed. VaR denotes a loss ℓ that will not be exceeded at certain confidence level $1 - p$ over a time horizon from t to $t + \Delta t$. To be more specific,

$$P(V_{t+\Delta t} - V_t < \ell) = p,$$

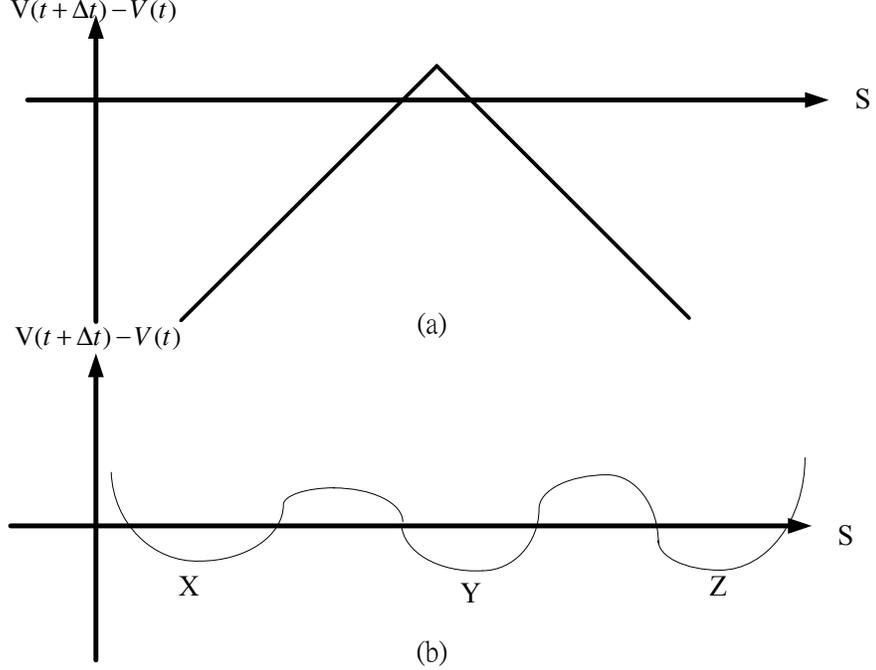
where V_τ denotes portfolio value at time τ . Typically, p is close to zero. For convenience, we define $V_{t+\Delta t} - V_t$ as the portfolio gain over the time span Δt .

The VaR for some simple cases can be computed analytically [2]. For example, if the “economic variables”, say the stock prices, that affect the portfolio value are assumed to follow certain simple diffusion processes like the lognormal diffusion process, then the distributions of the stock prices at a certain time point can be easily described. The VaR for holding the stocks can thus be computed analytically. In addition, the analytical formulas for some simple options like vanilla options can also be easily derived [1]. Thus the VaR for holding vanilla options can also be computed analytically. However, if the stock prices are assumed to follow some complex diffusion processes like the jump diffusion process, it would be difficult or even impossible to describe the distributions of stock prices. Thus the analytical formula for VaR is hard to derive.

The Monte Carlo simulation is a flexible and powerful tool to estimate VaR since it is usually more easily to sample the stock prices from complex diffusion price processes than to estimate the distributions of stock prices at a certain time point. We can repeatedly reevaluate the portfolio value by the sample stock prices and the distribution of the portfolio loss may be estimated. However, estimating VaR by the Monte Carlo simulation can be very inefficient since the event that the portfolio loss exceeds ℓ is rare (note that p is close to zero) and a large number of samples is thus required to obtain an accurate probability estimate of this rare event. Glasserman *et al.* develop variance reduction techniques based on importance sampling that can drastically reduce the number of samples required to achieve accurate estimates of rare events [4, 5]. In their method, the stock prices are sampled from a new probability measure where the event of significant loss is more likely to happen than in the original one. This new probability measure is selected to “asymptotically minimize” the second moment of the estimator for estimating $P(V_{t+\Delta t} - V_t < \ell)$. (Details will be introduced in Section 2.) However, their technique might perform poorly for some complex portfolios like shorting straddle options and multiple-minima portfolio (see Fig. 1). Take shorting straddle options for example. Shorting straddle options suffers significant loss when there is a significant increase or decrease in the underlying stock price and tilting the probability to make one huge-loss event, says a significant decrease in the stock price, more likely to happen will make the other event (a significant increase in the stock price) much more rare. The numerical results in our paper shows that the Glasserman’s approach is less efficient than the naive Monte Carlo simulation under such circumstance since it takes Glasserman’s approach much more samples to accurately estimate the probability of the event of significant increase in the stock price. Glasserman *et al.* argue that this problem can be solved by the delta-gamma approximation [6, 8] if the portfolio gain can be well approximated by a quadratic function of the stock price. However, it can be observed in Fig. 1 that the portfolio gain of shorting straddle options near the option maturity date can not be well approximated by a quadratic function of the stock price. It is also hopeless to approximate the portfolio gain of the three-minima portfolio by a quadratic function. Thus the research on improving the performance of estimating the VaR by importance sampling has not been satisfactorily settled.

The major contribution of this paper is the hybrid importance sampling algorithm that can solve the aforementioned problem. The hybrid importance sampling algorithm is composed of some sub simulations; each sub simulation focuses on one significant loss (of the portfolio value) event. For example, our algorithm for estimating the VaR for shorting straddle options can be decomposed into two sub simulations. One focuses on the significant decrease in the stock price and the other focuses on the significant increase in the stock price. The algorithm for estimating the VaR for the three-minima portfolio can be decomposed into three sub simulations. These three sub simulations focus on huge-loss events X, Y, and Z, respectively. Each sub simulation tilts its probability measure of the stock price to “asymptotically minimize” the second moment for estimating the probability of the huge-loss event focused by that sub simulation. Decomposing a Monte Carlo simulation into some sub simulations results in a new problem: How much computational time should be allocated to each sub simulation to minimize the second moment of the estimation result given a limit on computational time? To solve the problem, we use the number of stock price samples as a proxy for computational time. The allotment of stock price samples to each sub simulation is then properly determined by Lagrange’s multiplier to achieve our goal. To keep simplicity, our paper will demonstrate how we apply the hybrid importance sampling algorithm to estimate the VaR of shorting straddle options by assuming that the stock price follows the complex Merton’s jump diffusion process [7]. Numerical experiments are given to verify the superiority of our algorithm.

Fig. 1. The Relationship between the Stock Price and the Portfolio Gain.



The x - and y -axis denote the stock price and the portfolio gain, respectively. Panel (a) denotes the case of shorting straddle options near the option maturity date. Panel (b) denotes the case of three-minima portfolio mentioned in [2]. X , Y , and Z denotes there huge-loss events of this portfolio.

2 Preliminaries

2.1 The Stock Price Process

Define S_t as the stock price at year t . Under the Merton's jump diffusion model, the stock price process can be expressed as

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + (e^X - 1)dN_t \quad (1)$$

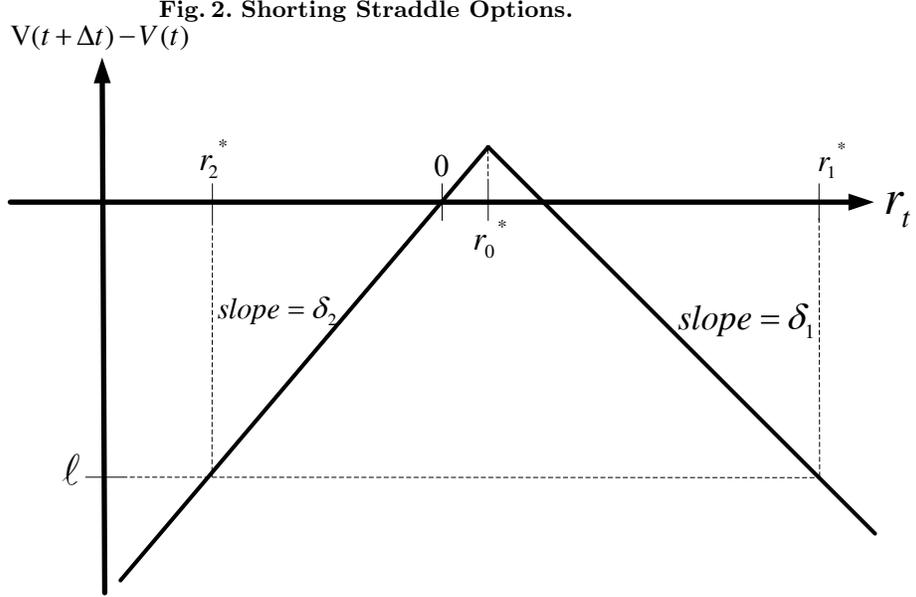
where W_t is the standard Wiener process, μ is the average stock return per annum, σ is the annual volatility, X is a normal random variable that model the jump size, and N_t denotes the Poisson process. We further assume that $X \sim N(\eta, \delta^2)$ and $P(dN_t = 1) = \lambda dt$. Note that Eq. (1) degenerates into the Black-Scholes lognormal diffusion process when $\lambda = 0$. Define the stock return over the time horizon Δt as follows:

$$r_t \equiv \frac{S_{t+\Delta t} - S_t}{S_t} \approx \mu \Delta t + \sigma \sqrt{\Delta t} Z + \sum_{i=1}^{N_{\Delta t}} Z_i, \quad (2)$$

where $Z \sim N(0, 1)$, $N_{\Delta t}$ denotes the number of jumps between time t and $t + \Delta t$, $Z_i \sim N(\eta, \delta^2)$.

2.2 Straddle Options

Stock options are derivative securities that give their buyer the right, but not the obligation, to buy or sell the underlying stocks for a contractual price called the exercise price K at maturity. Assume that the options mature at time $t + \Delta t$, then the payoffs of a call option and a put option at maturity are $\max(S_{t+\Delta t} - K, 0)$ and $\max(K - S_{t+\Delta t}, 0)$, respectively. Shorting straddle options denotes a short position in δ_1 units call



The x - and y -axis denote the stock return and the portfolio gain, respectively.

options and δ_2 units put options with the same strike price. The payout (or the portfolio loss) for shorting these options at maturity is therefore $\delta_1 \max(S_{t+\Delta t} - K, 0) + \delta_2 \max(K - S_{t+\Delta t}, 0)$. We interpret the portfolio gain in terms of stock return in Fig. 2. Note that

$$r_0^* \equiv \frac{K - S_t}{S_t}. \quad (3)$$

The portfolio gain can be expressed as follows:

$$\text{Portfolio Gain} = \begin{cases} r_0^* S_t \delta_2 + (r_t - r_0^*) S_t \delta_1, & \text{if } r_t \geq r_0^*, \\ r_t S_t \delta_2, & \text{if } r_t < r_0^*. \end{cases} \quad (4)$$

The portfolio loss exceeds ℓ if the stock returns is larger than r_1^* or lower than r_2^* , where $r_1^* = \frac{\ell - r_0^* S_t \delta_2 + r_0^* S_t \delta_1}{S_t \delta_1}$ and $r_2^* = \frac{\ell}{S_t \delta_2}$.

2.3 Glasserman's Importance Sampling Approach

A brief example is given to demonstrate the importance sampling approach of Glasserman *et al.* [4]. The stock price process stated in this subsection follows Black-Scholes long-normal price process (i.e. $N_{\Delta t} = 0$ in Eq. (2)). We consider a simple portfolio that contains only a stock. Note that the portfolio loss exceeds ℓ when the stock return is less than $-r_p$ ($\equiv \frac{-\ell}{S_t}$). Then the event A that denotes the the portfolio loss larger than ℓ is

$$A \equiv \{Z : f(Z) \equiv -r_t - r_p = -\mu \Delta t - \sigma \sqrt{\Delta t} Z - r_p > 0\}.$$

To improve the performance for estimating the probability of event A , the random variable Z is sampled from a new probability measure \mathbb{P}_θ instead of the original probability measure \mathbb{P} . The likelihood rate for these two probabilities measures is

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = \exp\{\theta f(Z) - \Psi(\theta)\}, \quad (5)$$

where $\Psi(\theta) \equiv \log E[\exp(\theta f(Z))]$. Define $E_{\mathbb{P}_\theta}$ as the expected value measured under \mathbb{P}_θ and

$$A_\theta \equiv \{Z_\theta : -\mu\Delta t - \sigma\sqrt{\Delta t}Z_\theta - r_p > 0\},$$

where $Z_\theta \sim N(\theta\sigma\sqrt{\Delta t}, 1)$. Then we have

$$p = E(1_A) = E_{\mathbb{P}_\theta}[1_{A_\theta} \exp(-\theta f(Z_\theta) + \Psi(\theta))].$$

That is, $1_{A_\theta} \exp(-\theta f(Z_\theta) + \Psi(\theta))$ is an unbiased estimator of p under probability measure \mathbb{P}_θ . The second moment of the estimator is then

$$\text{Second moment} = E_{\mathbb{P}_\theta}[1_{A_\theta} \exp(-2\theta f(r_t) + 2\Psi(\theta))] \leq \exp(\Psi(\theta)). \quad (6)$$

To asymptotically optimize the performance of the Monte Carlo simulation, a proper θ is selected to minimize $\exp(\Psi(\theta))$ by the following equation:

$$\Psi'(\theta) = 0. \quad (7)$$

\mathbb{P}_θ is then determined by substituting θ (obtained from Eq. (7)) into Eq. (5).

3 The Hybrid Importance Sampling Algorithm

The hybrid importance sampling algorithm is developed to estimate VaR for a financial portfolio with multiple huge-loss events. It consists of some sub simulations and each sub simulation focuses on one huge-loss event and tilts its probability measure to asymptotically minimize variance for estimating the probability of that event by Eqs. (5)–(7). We will use the portfolio of shorting straddle options illustrated in Fig. 2 to show how the hybrid importance sampling algorithm works. Note that the stock price process is assumed to follow Merton's jump diffusion process.

It can be observed in Fig. 2 that the loss of shorting straddle options exceeds ℓ when the stock return r_t exceeds threshold r_1^* or is below r_2^* . For convenience, we use event A_1 and A_2 to denote the events $\{r_t > r_1^*\}$ and $\{r_t < r_2^*\}$, respectively, as follows:

$$\begin{aligned} A_1 &\equiv \{r_t : f_1(r_t) \equiv -r_0^*\delta_2 - (r_t - r_0^*)\delta_1 - r^* > 0\}, \\ A_2 &\equiv \{r_t : f_2(r_t) \equiv -r_t\delta_2 - r^* > 0\}, \end{aligned} \quad (8)$$

where $r^* = -\ell/S_t$ and formula $f_1(r_t)$ and $f_2(r_t)$ are derived from Eq. (4). Note that A_1 and A_2 are mutually exclusive. Then we have

$$p = E[1_{A_1 \cup A_2}] = E[1_{A_1}] + E[1_{A_2}].$$

In other words, the Monte Carlo simulation for estimating p can be decomposed into two sub simulations; the first simulation focuses on estimating the probability of event A_1 , and the second one focuses on event A_2 .

Next we describe how to efficiently estimate $E[1_{A_1}]$ and $E[1_{A_2}]$ by importance sampling. Assume that the sub simulations for estimating $E[1_{A_1}]$ and $E[1_{A_2}]$ tilt their probabilities from \mathbb{P} to \mathbb{P}_{θ_1} and \mathbb{P}_{θ_2} , respectively. Then θ_1 and θ_2 are derived as follows: Define $\Psi_1(\theta_1) \equiv \log E[\exp(\theta_1 f_1(r_t))]$ and $\Psi_2(\theta_2) \equiv \log E[\exp(\theta_2 f_2(r_t))]$. We first calculate $E[\exp(\theta_1 f_1(r_t))]$ as follows:

$$\begin{aligned} E[\exp(\theta_1 f_1(r_t))] &= E[\exp(\theta_1(-r_0^*\delta_2 - r_t\delta_1 + r_0^*\delta_1 - r^*))] \\ &= \exp(-\theta_1 r_0^*\delta_2 + \theta_1 r_0^*\delta_1 - \theta_1 r^*) E(\exp(-\theta_1 r_t\delta_1)). \end{aligned}$$

Note that

$$\begin{aligned} E(\exp(-\theta_1 r_t \delta_1)) &= E \left[\exp \left(-\theta_1 \delta_1 \left(\mu \Delta t + \sigma \sqrt{\Delta t} Z + \sum_{i=1}^{N_{\Delta t}} Z_i \right) \right) \right] \\ &= \exp(-\theta_1 \delta_1 \mu \Delta t + 0.5 \sigma^2 \Delta t \theta_1^2 \delta_1^2) E \left[\exp \left(-\theta_1 \delta_1 \sum_{i=1}^{N_{\Delta t}} Z_i \right) \right], \end{aligned}$$

and

$$\begin{aligned} E \left[\exp \left(-\theta_1 \delta_1 \sum_{i=1}^{N_{\Delta t}} Z_i \right) \right] &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t^n)}{n!} E \left[\exp \left(-\theta_1 \delta_1 \sum_{i=1}^n Z_i \right) \right] \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t^n)}{n!} \exp(-n \theta_1 \delta_1 \eta + 0.5 n \theta_1^2 \delta_1^2 \delta^2) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t \exp(-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2))^n}{n!} \\ &= \exp(\lambda_t e^{-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2} - \lambda_t), \end{aligned}$$

where $\lambda_t \equiv \lambda \Delta t$. Thus θ_1 can be obtained by numerically solving the equation $\Psi_1'(\theta_1) = 0$ (see Eq. (7)) as follows:

$$\Psi_1'(\theta_1) = -r_0^* \delta_2 + r_0^* \delta_1 - r^* - \delta_1 \mu \Delta t + \sigma^2 \Delta t \theta_1 \delta_1^2 + \lambda_t (-\delta_1 \eta + \theta_1 \delta_1^2 \delta^2) \exp(-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2) = 0.$$

Similarly, it can be derived that

$$E[\exp(\theta_2 f_2(r_t))] = \exp\left(-\theta_2 \delta_2 \mu \Delta t + 0.5 \sigma^2 \Delta t \theta_2^2 \delta_2^2 - \theta_2 r^* + \lambda_t e^{-\theta_2 \eta \delta_2 + 0.5 \theta_2^2 \delta_2^2 \delta^2} - \lambda_t\right).$$

θ_2 can also be solved numerically by the equation $\Psi_2'(\theta_2) = 0$ as follows:

$$\Psi_2'(\theta_2) = -\delta_2 \mu \Delta t + \sigma^2 \Delta t \theta_2 \delta_2^2 - r^* + \lambda_t (-\eta \delta_2 + \theta_2 \delta_2^2 \delta^2) e^{-\theta_2 \eta \delta_2 + 0.5 \theta_2^2 \delta_2^2 \delta^2} = 0.$$

Finally, the new probability distribution \mathbb{P}_{θ_1} sampled by the first sub simulation can be derived by Eq. (5) as follows:

$$\begin{aligned} d\mathbb{P}_{\theta_1} &= d\mathbb{P} \exp(\theta_1 f_1(r_t) - \Psi_1(\theta_1)) \tag{9} \\ &= \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda_t} \lambda_t^n}{n!} \left(\frac{1}{\sqrt{2\pi\delta^2}} \right)^n e^{-\frac{\sum_{k=1}^n (Z_k - \eta)^2}{2\delta^2}} \right] \exp(\theta_1 f_1(r_t) - \Psi_1(\theta_1)) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z + \theta_1 \delta_1 \sigma \sqrt{\Delta t})^2}{2}} \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda_t e^{-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2}} (\lambda_t e^{-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2})^n}{n!} \left(\frac{1}{\sqrt{2\pi\delta^2}} \right)^n e^{-\frac{\sum_{k=1}^n (Z_k - (\eta - \theta_1 \delta_1 \delta^2))^2}{2\delta^2}} \right]. \end{aligned}$$

That is, the first sub simulation tilts the probability from \mathbb{P} to \mathbb{P}_{θ_1} , where the distributions of random variables (in Eq. (2)) are changed as follows: $Z \sim N(-\theta_1 \delta_1 \sigma \sqrt{\Delta t}, 1)$, $N_{\Delta t} \sim \text{Poisson}(\lambda_t e^{-\theta_1 \delta_1 \eta + 0.5 \theta_1^2 \delta_1^2 \delta^2})$ and $Z_i \sim N(\eta - \theta_1 \delta_1 \delta^2, \delta^2)$. Note that

$$E[1_{A_1}] = E_{\mathbb{P}_{\theta_1}} [1_{A_1} \exp(-\theta_1 f_1(r_t) + \Psi_1(\theta_1))].$$

Thus $E[1_{A_1}]$ is estimated by sampling the unbiased estimator $1_{A_1} \exp(-\theta_1 f_1(r_t) + \Psi_1(\theta_1))$ from \mathbb{P}_{θ_1} in the first sub simulation. Similarly, the probability distribution \mathbb{P}_{θ_2} used by the second sub simulation can be derived as

$$\begin{aligned} d\mathbb{P}_{\theta_2} &= d\mathbb{P} \exp(\theta_2 f_2(r_t) - \Psi_2(\theta_2)) \tag{10} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z + \theta_2 \delta_2 \sigma \sqrt{\Delta t})^2}{2}} \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda_t e^{-\theta_2 \delta_2 \eta + 0.5 \theta_2^2 \delta_2^2 \delta^2}} (\lambda_t e^{-\theta_2 \delta_2 \eta + 0.5 \theta_2^2 \delta_2^2 \delta^2})^n}{n!} \left(\frac{1}{\sqrt{2\pi\delta^2}} \right)^n e^{-\frac{\sum_{k=1}^n (Z_k - (\eta - \theta_2 \delta_2 \delta^2))^2}{2\delta^2}} \right]. \end{aligned}$$

Note also that

$$E[1_{A_2}] = E_{\mathbb{P}_{\theta_2}} [1_{A_2} \exp(-\theta_2 f_2(r_t) + \Psi_2(\theta_2))].$$

The second sub simulation estimates $E[1_{A_2}]$ by sampling the unbiased estimator $1_{A_2} \exp(-\theta_2 f_2(r_t) + \Psi_2(\theta_2))$ from \mathbb{P}_{θ_2} .

To minimize the second moment for estimating p given a constraint on computational time, an optimal way to allocate computational time to each simulation is developed as follows: The number of stock return samples serves as a proxy of computational time. Assume that we can only sample \mathbb{N} stock returns, and the numbers of stock returns samples in the first and the second simulations are n_1 and n_2 , respectively. By Eq. (6), the upper bound of the second moment of the estimator $1_{A_1} \exp(-\theta_1 f_1(r_t) + \Psi_1(\theta_1))$ (or $1_{A_2} \exp(-\theta_2 f_2(r_t) + \Psi_2(\theta_2))$) under the probability measure \mathbb{P}_{θ_1} (or \mathbb{P}_{θ_2}) is $\exp(\Psi_1(\theta_1))$ (or $\exp(\Psi_2(\theta_2))$). The second moment for estimating p is then

$$\frac{\exp(\Psi_1(\theta_1))}{n_1} + \frac{\exp(\Psi_2(\theta_2))}{n_2}. \quad (11)$$

To minimize Eq. (11) under the constraint $n_1 + n_2 = \mathbb{N}$, n_1 and n_2 can be solved by Lagrange multiplier as follows:

$$n_1 = \frac{\sqrt{\exp(\Psi_1(\theta_1))}}{\sqrt{\exp(\Psi_1(\theta_1))} + \sqrt{\exp(\Psi_2(\theta_2))}} \mathbb{N} \quad (12)$$

$$\frac{\sqrt{\exp(\Psi_2(\theta_2))}}{\sqrt{\exp(\Psi_1(\theta_1))} + \sqrt{\exp(\Psi_2(\theta_2))}} \mathbb{N} \quad (13)$$

4 Numerical Results

Numerical experimental results are given in Table 1 and 2 to verify the superiority of the hybrid importance sampling algorithm, where the stock price processes in Table 1 and 2 are assumed to follow lognormal diffusion process and Merton's jump diffusion process, respectively. We do 100 estimations for each Monte Carlo simulation method and each estimation is based on 10000 stock price samples. The first column in Table 1 and 2 denotes the probability measure of the stock price return sampled by each Monte Carlo simulation method, where \mathbb{P} denotes the original stock price return probability measure defined in Eq. (2), \mathbb{P}_{θ_1} denotes the probability measure defined in Eq. (9), and \mathbb{P}_{θ_2} denotes the probability measure defined in Eq. (10). **Hybrid** denotes the hybrid importance sampling algorithm that is composed of two sub simulations, which sample the stock price return from \mathbb{P}_{θ_1} and \mathbb{P}_{θ_2} , respectively. The numbers of samples allocated to these two simulations are determined in Eq. (12) and (13), respectively. The second column in Table 1 and 2 denotes the average of the estimation results for each Monte Carlo simulation, and the third column denotes the variance of the estimation results for each Monte Carlo simulation.

We first focus on Table 1. The event that the portfolio loss exceeds ℓ is about 0.0349. This event can be decomposed into two mutually exclusive events A_1 and A_2 (see Eq. (8)). The event A_1 (with probability $P(A_1) \approx 0.0049$) is less likely to happen than the event A_2 (with probability $P(A_2) \approx 0.0299$). Although tilting the probability measure of the stock return from \mathbb{P} to \mathbb{P}_{θ_1} makes the Monte Carlo simulation estimate $P(A_1)$ more efficiently and accurately, it also damages the accuracy for estimating $P(A_2)$. It can be observed that this tilting produce inaccurate probability estimation (0.0139) with large variance. Similar problem applies to the Monte Carlo simulation that tilts the probability measure to \mathbb{P}_{θ_2} ; this Monte Carlo simulation is inadequate to estimate $P(A_1)$. But tilting the probability measure to \mathbb{P}_{θ_2} is better than tilting the probability to \mathbb{P}_{θ_1} since the event A_2 constitutes the bulk of the of the event that the portfolio loss exceeds ℓ . Note that both important sampling methods mentioned above are less efficient than the primitive Monte Carlo

Table 1. Estimating p under the Lognormal Stock Price Process Model.

Probability Measure	\mathbb{P}	\mathbb{P}_{θ_1}	\mathbb{P}_{θ_2}	Hybrid
\hat{p}	0.0349	0.0139	0.0395	0.0349
$\text{Var}(\hat{p})$	3.55×10^{-6}	4.14×10^{-3}	2.52×10^{-3}	2.28×10^{-7}

The numerical settings are listed as follows: The stock average annual return μ is 0.05, the volatility of the stock price σ is 0.3, the time span Δt is 0.008 year, r_0^* defined in Eq. (3) is 0.01, and r^* is 0.05. The portfolio contains a short position in 1 unit call option ($\delta_1 = -1$) and 1 unit put option ($\delta_2 = 1$). Note that $\lambda = 0$ in this case.

Table 2. Estimating p under the Merton's Jump Diffusion Stock Price Process Model.

Probability Measure	\mathbb{P}	\mathbb{P}_{θ_1}	\mathbb{P}_{θ_2}	Hybrid
\hat{p}	0.0405	0.0541	0.0426	0.0403
$\text{Var}(\hat{p})$	4.05×10^{-6}	1.97×10^{-2}	6.21×10^{-4}	5.44×10^{-7}

The numerical settings follow the settings listed in Table 1 except that $\lambda = 6$, the average jump size η is 0, and δ is 0.03.

simulation (that samples the stock return from \mathbb{P}). On the other hand, the hybrid importance sampling algorithm performs better than the primitive Monte Carlo simulation. It produces accurate estimation for p and reduces the variance to $1/15$ ($\approx \frac{2.28 \times 10^{-7}}{3.55 \times 10^{-6}}$) of the variance of primitive Monte Carlo simulation. In other words, the sample size of the primitive Monte Carlo simulation should be 15 times the sample size of the hybrid importance sampling algorithm to make the former method achieve the same accuracy level as the latter method.

The stock price process is assumed to follow the Merton's jump diffusion process in Table 2. The probability of the event that the portfolio loss exceeds ℓ is larger in this case since the jumps in stock price makes the events A_1 and A_2 more likely to happen. Naively tilting the probability measure of the stock return to \mathbb{P}_{θ_1} (or \mathbb{P}_{θ_2}) also performs poorly in this case. Note that the hybrid importance sampling algorithm still outperforms the primitive Monte Carlo simulation by reducing the variance to $1/7.5$ ($\approx \frac{5.44 \times 10^{-7}}{4.05 \times 10^{-6}}$) of the variance of primitive Monte Carlo simulation.

5 Conclusion

This paper suggests a new Monte Carlo simulation, the hybrid importance sampling algorithm, that can efficiently estimate the VaR of complex financial portfolios. Our approach is composed of some sub simulations; each sub simulation focus on one huge-loss event. The performance of each sub simulation can be improved by importance sampling. To minimize the variance for estimating VaR given a constraint on computational time, we use the number of stock price samples as a proxy of computational time. The allotment of stock price samples to each sub simulations is then determined by Lagrange multiplier. Numerical results given in this paper verify the superiority of our method.

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