Using the LIBOR Market Model to Price the Interest Rate Derivatives：A Recombining Binomial Tree Methodology

以 LIBOR 市場模型評價利率衍生性商品：結合節點二項樹方法

Abstract

LIBOR market model (LMM) is a complicate interest rate model and it is hard to be implemented by the numerical methods. Because of the non-Markov property of the LMM, the size of all naive lattice models will grow explosively and would not be evaluated by computers. This thesis proposes a recombining LMM lattice model by taking advantages of lattice construction methodology proposed by Ho, Stapleton, and Subrahmanyam (HSS). We first rewrite the discrete mathematical models for LMM suggested by Poon and Stapleton. Then we derive the conditional means and the variances of the discrete forward rates which are important for the tree construction. Finally, using the construction methodology proposed by HSS we build our pricing model for the interest rate derivatives. Numerical results are given in Chapter 5 suggest that our lattice method can produce convergent and accurate pricing results for interest rate derivatives.

Keywords: HSS, LMM, bond option, caplet

摘 要

LIBOR 市場模型是一個複雜的利率模型，難以利用數值方法計算之，加上該模型具有非馬可夫的特性，導致在學術研究上使用樹狀結構評價方法來評價商品時，會出現節點數呈現爆炸性成長的現象，而此成長現象甚至遠超過電腦所能計算的範圍。因此，本文的研究宗旨在於採用 Ho、Stapleton 和 Subrahmanyam(HSS) 建造樹狀評價結構模型的特點，提出一個嶄新的節點重合的遠期利率樹狀結構 LIBOR 市場模型來評價利率衍生性商品。首先，本研究參考 Poon 和 Stapleton 所建議的方法，將連續型的 LIBOR 市場模型改寫成離散模型；其次，藉由此離散的遠期模型導出建造遠期利率樹狀結構所需要的重要參數—條件期望值和條件變異數後，再利用 HSS 建構樹狀結構的特點來建造遠期利率樹狀評價模型。最後，利用本研究所提出的遠期利率樹狀模型來評價利率衍生性商品，其結果顯示本研究的模型提供了一個準確且快速收斂的現象。

關鍵字：LIBOR 市場模型、結合節點樹狀結構評價法、遠期利率、債券選擇權、利率上限選擇權

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1 Introduction

Many traditional interest rate models are based on instantaneous short rates and instantaneous forward rates. However, these rates are unobservable in the daily markets, in results, these models are hard to calibrate in the daily markets. Therefore, the most commonly used model in practice is the LIBOR market model (LMM). LIBOR market model is based on the forward LIBOR rate observed from daily markets. The model was first proposed by Brace, Gatarek and Musiella (1997) (abbreviate as BGM). In their assumptions, the LIBOR rate follows the lognormal distribution, which derives the theoretical pricing formula for the caplet that consistent with the Black's model (1976).

However, when implementing the LMM by the lattice method, the tree grows explosively since nodes of the tree are not recombining. That is, LMM has the non-Markov property as well as HJM model. The non-recombining phenomenon of the nodes make our lattice method inefficient and difficult to price. To address this problem, this research adapts the HSS methodology proposed by Ho, Stapleton, and Subrahmanyam (1995) to construct a recombining binomial tree for LMM. By applying the HSS methodology into the LMM, the lattice valuation method becomes feasible in pricing the interest rate derivatives.

The lattice method we proposed here makes us have not to rely on the Monte Carlo simulation because our tree-based method is more accurate and efficient. Besides, the lattice method can take American-style features, such as early exercise or early redemption, which is an intractable problem in Monte Carlo simulation.

For the following thesis in description, Chapter 2 reviews some important interest rate models. Chapter 3 introduces the market conventions about LMM and derives the drift of discrete-time version of LMM which follows the development in

Poon and Stapleton (2005). In chapter 4, we introduce the HSS recombining node methodology (1995) into the discrete-time version of LMM which derived in chapter 3 and construct the pricing model. In chapter 5, we apply our proposed model to price the value of bond option and the caplet. Besides computing the derivatives numerical price, we also compare the prices computed by our model with the Black-model caplet prices. At last in chapter 6 concludes my work and make possible suggestions for the future work.

2 Review of Interest Rate Models

In this section we introduce some important interest rate models which can categorize into two different models: equilibrium models and no-arbitrage models. The no-arbitrage models can be further classified into three parts: instantaneous short rate models, instantaneous forward rate models and forward rate models.

2.1 Equilibrium Models

Equilibrium models are under the assumptions about economic variables derives from a process for the short rate *r*. The short rate *r* is governed by stochastic process, like geometric Brownian motion, and has the characteristic of *mean reversion*. In other words, interest rates appear to be pulled back to some long-run average level and this phenomenon is known as *mean reversion*. This section introduces some of these models that have aforementioned properties.

Vasicek Model

In Vasicek model, interest rate *r* is supposed to follow the Ornstein-Uhlenbeck process and has the following expression under the risk-neutral measure:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t)$$

where mean reversion rate α, reversion level β, and volatility σ are constants. But its short is that interest rate could be negative due to the stochastic term $dW(t)$ is normally distributed. In this model, Vasicek shows that the general pricing form of $P(t,T)$ which is the price at the time *t* of a zero coupon bond with principal \$1 maturing at time *T*:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T)=\frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T)=\exp[\frac{(B(t,T)-T+t)(\alpha^2\beta-\sigma^2/2)}{\alpha^2}-\frac{\sigma^2 B(t,T)^2}{4\alpha}]$$

CIR Model

To improve the drawback in Vasicek model, Cox, Ingersoll, and Ross have proposed an alternative model which make rate r always non-negative. Under the risk neutral measure, the CIR model follows the following process:

$$dr(t)=\alpha(\beta-r(t))dt+\sigma\sqrt{r(t)}dW(t)$$

which has the same mean-reverting drift as Vasicek model. However, CIR model use the square root of rate r to replace the constant volatility in Vasicek model that makes the model have a non-central chi-squared distribution. Besides, CIR model has the same general form of bond prices as in Vasicek model.

$$P(t,T)=A(t,T)e^{-B(t,T)r(t)}$$

but its $A(t,T)$ and $B(t,T)$ are different:

$$B(t,T)=\frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma}$$

$$A(t,T)=[\frac{2\gamma e^{(\alpha+\gamma)(T-t)/2}}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma}]^{2\alpha\beta/\sigma^2}$$

where $\gamma=\sqrt{\alpha^2+2\sigma^2}$.

2.2 No-Arbitrage Models

Although equilibrium models have the mean-reverting property, it doesn't fit today's term structure of interest rates. Thus, no-arbitrage model is designed to calibrate today's term structure of interest rates. Furthermore, in no-arbitrage model, today's term structure of interest rate is an input and the drift of the short rate is generally time dependent.

2.2.1 Instantaneous Short Rate Models

Ho-Lee Model

The first model of no-arbitrage models is Ho-Lee model, which is shown under the risk-neutral measure:

$$dr(t) = \theta(t)dt + \sigma dW(t)$$

where *θ (t)* is a function of time chosen to ensure that the model fits the initial term structure, and it is relative to the instantaneous forward rate. The relevance to instantaneous forward rate is

$$\theta(t) = f_t(0,t) + \sigma^2 t$$

where $f_t(0,t)$ is the instantaneous forward rate for maturity t as seen at time zero and the subscript *t* denotes a partial derivative with respect to *t.* Moreover, the price of the zero coupon bond at time t can be expressed as

$$P(t,T) = A(t,T)e^{-r(t)(T-t)}$$

where

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} + (T-t)f(0,t) - \frac{1}{2}\sigma^2 t(T-t)^2.$$

Hull-White Model

The other no-arbitrage model is Hull-White model, which is a simple but powerful model. It is a generalization of the Vasicek model and it provides an exact fit to the initial term structure. The model is shown under the risk-neutral measure:

$$dr(t) = [\theta(t) - \alpha r(t)]dt + \sigma dW(t)$$

where α and σ are constants and the function of $\theta(t)$ can be calculated from the initial term structure:

$$\theta(t) = f_t(0,t) + \alpha f(0,t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

Because the Hull-White model is the general form of the Vasicek model, it has the same general form of bond prices as in Vasicek model:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} + B(t,T)f(0,t) - \frac{1}{4\alpha^3}\sigma^2(e^{-\alpha T} - e^{-\alpha t})^2(e^{2\alpha t} - 1)$$

2.2.2 Instantaneous Forward Rate Model

HJM Model

Heath, Jarrow and Morton (1992) published an important paper described the evolution of the entire yield curve in continuous time. They proposed the dynamic form of the instantaneous forward rate and derived the stochastic process of the instantaneous forward rate $f(t,T)$ for the fixed maturity T under risk-neutral measure. The form is described as follows:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where

$W(t) = (W_1(t),\cdots,W_d(t))$ is a d-dimensional Brownian motion,

$\sigma(t,T) = (\sigma_1(t,T),\cdots,\sigma_d(t,T))$ is a vector of adapted processes,

$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds = \sum_{i=1}^{d}\sigma_i(t,T)\int_t^T \sigma_i(t,s)ds$.

Given the dynamics of the instantaneous forward rate $f(t,T)$, we can use the Ito's lemma to obtain the dynamics of the zero-coupon bond price $P(t,T)$:

$$dP(t,T) = P(t,T)[r(t)dt - (\int_t^T \sigma(t,s))dW(t)]$$

where $r(t)$ is the instantaneous short term interest rate at time *t,* that is

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(u,t) \int_u^t \sigma(u,s) ds du + \int_0^t \sigma(s,t) dW(s)$$

From the above formula, the short rate $r(t)$ in the HJM model is non-Markov and makes the tree of nodes construct to be non-recombined. However, one drawback of the HJM model is that it is expressed in terms of instantaneous forward rates, which are not directly observable in the market and difficult to calibrate the model to price the actively traded instruments. Therefore, a new model is developed to improve the aforementioned insufficiency.

2.2.3 Forward Rate Model

LIBOR Market Model (LMM)

The new alternative model, LIBOR market model (LMM), was discovered by Brace, Gatarek, and Musiela (1997) and was initially referred to as the BGM model by practitioners. However, Miltersen, Sandmann, and Sondermann (1997) discovered this model independently, and Jamshidian (1997) also contributed significantly to its initial development. To reflect the contribution of multiple authors, many practitioners, including Rebonato(2002), renamed this model to LIBOR market model.

There are two common versions of the LMM, one is the lognormal forward LIBOR model (LFM) for pricing caps and the other is the lognormal swap model (LSM) for pricing swaptions. The LFM assumes that the discrete forward LIBOR rate follows a lognormal distribution under its own numeraire, while the LSM assumes that the discrete forward swap rate follows a lognormal distribution under the swap numeraire. The two assumptions do not match theoretically, but lead to small discrepancies in calibrations using realistic parameterizations. The following derivations are based on the LFM.

The LFM specifies the discrete forward rate $f(t;T_i,T_{i+1})$ which is seen at time *t* during the period between time T_i and time T_{i+1} that is different from the instantaneous forward rate $f(t,T)$ as seen at time *t* for a contract maturing at time T and follows zero-drift stochastic process under its own forward measure:

$$\frac{df(t;T_i,T_{i+1})}{f(t;T_i,T_{i+1})} = \sigma_i(t)dW_i(t)$$

where $dW_i(t)$ is a Brownian motion under the forward measure $Q^{T_{i+1}}$ defined with respect to the numeraire asset $P(t,T_{i+1})$ and where $\sigma_i(t)$ measures the volatility of the forward rate process. Using Ito's lemma, the stochastic process of the logarithm of the forward rate is given as follows:

$$d\ln f(t;T_i,T_{i+1}) = \frac{-\sigma_i^2(t)}{2}dt + \sigma_i(t)dW_i(t) \qquad (2.3.1)$$

The stochastic integral of equation (2.3.1) can be given as follows. For all $0 \le t \le T_i$,

$$\ln f(t;T_i,T_{i+1}) = \ln f(0;T_i,T_{i+1}) - \int_0^t \frac{-\sigma_i^2(u)}{2}du + \int_0^t \sigma_i(u)dW_i(u) \qquad (2.3.2)$$

Since the volatility function $\sigma_i(t)$ is deterministic, the logarithm of forward rate is normally distributed, implying that the forward rate is lognormally distributed. For $t = T_i$, equation (2.3.2) implies that the future LIBOR rate $L(T_i,T_{i+1}) = f(T_i;T_i,T_{i+1})$ is also lognormally distributed. This explains why this model is called the lognormal forward LIBOR model. Thought each forward rate is lognormally distributed under its own forward measure, it is not lognormally distributed under other forward measure.

3 Market Conventions of the LMM and the Discrete-Time Version of the LMM

To enter the world of LMM, we have to be familiar with the terminologies and instruments that used by the market practitioners. We first introduce the basic terms and some instruments such like caplets and FRAs, and then use the important results in the Poon and Stapleton (2005) to derive the discrete-time version of the LMM.

3.1 Market Conventions of the LMM

The relationship between the discrete LIBOR rate $L(T_i, T_{i+1})$ for the term $\delta_i = T_{i+1} - T_i$ and the zero-coupon bond price $P(T_i, T_{i+1})$ is given as follows:

$$P(T_i, T_{i+1}) = \frac{1}{1 + \delta_i L(T_i, T_{i+1})} \tag{3.1.1}$$

where $t \le T_0 < T_1 < T_2 < \cdots < T_n$ is the time line and δ_i is called the tenor or accrual fraction for the period T_i to T_{i+1}.

The time t discrete forward rate for the term $\delta_i = T_{i+1} - T_i$ is related to the price ratio of two zero-coupon bonds maturing at times T_i and T_{i+1} as follows:

$$1 + \delta_i f(t; T_i, T_{i+1}) = \frac{P(t, T_i)}{P(t, T_{i+1})} \tag{3.1.2}$$

The forward rate converges to the future LIBOR rate at time T_i, or:

$$\lim_{\tau \to T_i} f(\tau; T_i, T_{i+1}) = L(T_i, T_{i+1}) \tag{3.1.3}$$

We can rewrite equation (3.1.2) as follows:

$$f(t; T_i, T_{i+1}) P(t, T_{i+1}) = \frac{1}{\delta_i} [P(t, T_i) - P(t, T_{i+1})]$$

Then, we define some of the basic terms we often used in the market and illustrate as follows:

$For(t,T_1,T_n)$: the forward price at time t to invest a zero coupon bond matured at time T_n at time T_1 and can be expressed as $P(t,T_n)/P(t,T_1)$.

$y(t,T_1)$: the annual yield rate at time t to time T_1 and its relation with the zero coupon bond is givens as $P(t,T_1)=1/(1+\delta_1 y(t,T_1))$.

$f(t;T_n,T_{n+1})$: the forward rate at time t for the time period T_n to T_{n+1} and its relation with forward price of a zero coupon bond is given as $For(t;T_n,T_{n+1})=1/(1+\delta_n f(t;T_n,T_{n+1}))$.

After introducing the basic terms, here, we introduce a popular interest rate option- an interest rate cap. A cap is composed of a series of caplets. For a T_i-maturity caplet, the practitioners widely use the Black's formula to obtain its value. Following is the Black's formula for the i-th caplet valued at time t:

$$caplet_i(t)=A\times\delta_i\times P(t,T_{i+1})[f(t;T_i,T_{i+1})N(d_1)-KN(d_2)] \quad (3.1.4)$$

where

$$d_1=\frac{\ln(f(t;T_i,T_{i+1})/K)+\sigma_i^2(T_i-t)/2}{\sigma_i\sqrt{T_i-t}},$$
$$d_2=\frac{\ln(f(t;T_i,T_{i+1})/K)-\sigma_i^2(T_i-t)/2}{\sigma_i\sqrt{T_i-t}},$$

A: the notional value of the caplet,

δ_i: the length of the interest rate reset interval as a proportion of a year,

$P(t,T_{i+1})$: the zero coupon bond price paying 1 unit at maturity date T_{i+1},

K: the caplet strike price,

σ_i: the Black implied volatility of the caplet,

$N(\cdot)$: the cumulative probability distribution function for a standardized normal distribution.

Furthermore, under the LIBOR basis, we can derive the same theoretical pricing equation for the caplet as equation (3.1.4) from the LFM model. Because both of LFM and Black's model are assuming that the forward rate follows the lognormal distribution and we get the consistent results.

Another instrument we illustrate here as a key to derive out the discrete-time version of the LMM is the forward rate agreement (FRA). A FRA is an agreement made at time *t* to exchange fixed-rate interest payments at a rate K for variable rate payments, on a notional amount A, for the loan period T_n to T_{n+1} equal to one year. The settlement amount at time T_n on a long FRA is

$$FRA(T_n) = \frac{A(y(T_n,T_{n+1}) - K)}{1 + y(T_n,T_{n+1})} \tag{3.1.5}$$

where $y(T_n,T_{n+1})$ is the annual yield at time T_n to T_{n+1}. At the time of the contract inception, a FRA is normally structured so that it has zero value. To avoid the arbitrage, the strike rate K is set equal to the market forward rate $f(t;T_n,T_{n+1})$. We denote the value of the FRA at time *t* as $FRA(t,T_n)$ which can be expressed as

$$FRA(t,T_n) = E_t[\frac{A(y(T_n,T_{n+1}) - f(t;T_n,T_{n+1}))}{1 + y(T_n,T_{n+1})}] = 0 \tag{3.1.6}$$

3.2 The Discrete-Time Version of the LMM

Now, we restate the most important results which are under the "risk neutral" measure in the Poon and Stapleton text (2005).

1. For a zero-coupon bond price is given by

$$P(t,T_n) = P(t,T_1)E_t(P(T_1,T_n)) \tag{3.2.1}$$

or we can write

$$E_t(P(T_1,T_n)) = \frac{P(t,T_n)}{P(t,T_1)} = For(t,T_1,T_n)$$

2. The drift of the forward bond price is given by

$$\begin{aligned} &E_t[For(T_1,T_i,T_n)] - For(t,T_i,T_n) \\ &\qquad = -\frac{P(t,T_1)}{P(t,T_n)}\operatorname{cov}_t[For(T_1,T_i,T_n),P(T_1,T_n)] \end{aligned} \tag{3.2.2}$$

3. The drift of T_n -period forward rate is obtained from the equation (3.1.6) and given by

$$\begin{aligned} &E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1}) = \\ &-\operatorname{cov}_t[f(T_1;T_n,T_{n+1}), \frac{1}{1+y(T_1,T_2)} \times \frac{1}{1+f(T_1;T_2,T_3)} \times \cdots \times \frac{1}{1+f(T_1;T_n,T_{n+1})}] \\ &\times (1+f(t;T_1,T_2)) \cdot (1+f(t;T_2,T_3)) \cdots\cdot (1+f(t;T_n,T_{n+1})) \end{aligned} \tag{3.2.3}$$

After restating the important results, we now apply the results to the LIBOR basis for the FRA and rewrite the equation (3.1.5) as follows

$$FRA(T_n) = \frac{A(f(T_n;T_n,T_{n+1}) - K)\cdot \delta_n}{1+\delta_n f(T_n;T_n,T_{n+1})} \tag{3.2.4}$$

where $\delta_n = T_{n+1} - T_n$ and we assume all the tenors are same (i.e. $\delta_1 = \delta_2 = \ldots = \delta_n = \delta$) and the notional amount A equal to one to make the equation briefer. And using the above results and similar steps to derive out the FRA value at time t of the equation (3.2.4) to generalize the T_n -maturity forward rate

$$\begin{aligned} &E_t[f(T_1;T_n,T_{n+1})] - f(t;T_n,T_{n+1}) = \\ &\frac{-1}{\delta}\operatorname{cov}_t[\delta f(T_1;T_n,T_{n+1}), \frac{1}{1+\delta f(T_1;T_1,T_2)} \cdots\cdot \frac{1}{1+\delta f(T_1;T_n,T_{n+1})}] \\ &\times (1+\delta f(t;T_1,T_2)) \cdot (1+\delta f(t;T_2,T_3)) \cdots\cdot (1+\delta f(t;T_n,T_{n+1})) \end{aligned} \tag{3.2.5}$$

We assume that the forward rate $f(T_1;T_n,T_{n+1})$ is the lognormal for all forward maturities, T_n. Then, we use the approximate result for the covariance term, that is for the small change around the value $X = a,\ Y = b$, we have $\operatorname{cov}(X,Y) \approx ab \operatorname{cov}(\ln X, \ln Y)$. Here we take $a = f(t;T_1,T_2)$ and $b = 1/(1+f(t;T_1,T_2))$ to evaluate $\operatorname{cov}_t(f(T_1;T_1,T_2), \frac{1}{1+f(T_1;T_1,T_2)})$, then we have

$$\operatorname{cov}_t(f(T_1;T_1,T_2), \frac{1}{1+f(T_1;T_1,T_2)}) =$$

$$f(t;T_1,T_2)(\frac{1}{1+f(t;T_1,T_2)}) \operatorname{cov}_t(\ln y(T_1,T_2), \ln \frac{1}{1+y(T_1,T_2)})$$

and substitute it into the equation (3.2.5) use the property of logarithms to express the drift of T_n-maturity forward rate as the sum of a series of covariance terms. Finally, to make our covariance terms in a recognizable form, we use the extension of Stein's lemma to evaluate the term with a form $\operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}), \ln(\frac{1}{1+f(T_1;T_1,T_2)}))$.

Stein's Lemma for lognormal variables

For joint-normal variables x and y

$$\operatorname{cov}(x, g(y)) = E(g'(y)) \cdot \operatorname{cov}(x,y)$$

Hence, if $x = \ln X$ and $y = \ln Y$, then

$$\operatorname{cov}(\ln X, \ln \frac{1}{1+Y}) = E(\frac{-Y}{1+Y}) \cdot \operatorname{cov}(\ln X, \ln Y)$$

Then we have

$$\operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}), \ln(\frac{1}{1+f(T_1;T_1,T_2)})) =$$

$$E_t(\frac{-f(T_1;T_1,T_2)}{1+f(T_1;T_1,T_2)}) \operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}), \ln f(T_1;T_1,T_2))$$

Here, we apply the result we mention above to the equation (3.2.5) and derive out the drift of the forward LIBOR rate as the sum of a series of covariance terms as follows:

$$
\begin{aligned}
&E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})= \\
&\quad f(t;T_n,T_{n+1})\times\frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)}\cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}),\ln f(T_1;T_1,T_2)] \\
&\quad +\cdots \\
&\quad + f(t;T_n,T_{n+1})\times\frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})}\cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}),\ln f(T_1;T_n,T_{n+1})]
\end{aligned}
\tag{3.2.6}
$$

We also assume that the covariance structure is inter-temporally stable and $\mathrm{cov}_t[\ln f(T_1;T_i,T_{i+1}),\ln f(T_1;T_n,T_{n+1})]$ is a function of the forward maturities and is not dependent on *t*. Then we define

$$
\mathrm{cov}_t[\ln f(T_1;T_i,T_{i+1}),\ln f(T_1;T_n,T_{n+1})]\equiv\tilde{\sigma}_{i,n} \qquad i=1,\ 2,\cdots,n
$$

where $\tilde{\sigma}_{i,n}$ is the covariance of the log i-period forward LIBOR and the log n-period forward LIBOR. Finally, we can rewrite equation (3.2.6) as follows:

$$
\begin{aligned}
\frac{E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})}{f(t;T_n,T_{n+1})}=&\frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n}+\frac{\delta f(t;T_2,T_3)}{1+\delta f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n} \\
&+\cdots+\frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n}
\end{aligned}
\tag{3.2.7}
$$

4 Introducing the HSS Recombining Node Methodology and Applying to The LIBOR Market Model

Ho, Stapleton, and Subrahmanyam (1995) suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process. Our assumption about the LMM satisfies the required conditions of the HSS methodology. Therefore, we apply the HSS methodology to construct the recombining trees for LMM. Now, we introduce the HSS methodology first and then apply it in the LMM.

4.1 The HSS Methodology

The HSS methodology assumes the price of underlying asset X follows a lognormal diffusion process:

$$d \ln X(t) = \mu(X(t), t)dt + \sigma(t)dW(t) \quad (4.1.1)$$

where μ and σ are the instantaneous drift and volatility of $\ln X$, and $dW(t)$ is a standard Brownian motion. They denote the unconditional mean at time 0 of the logarithmic asset return at time t_i as μ_i. The conditional volatility over the period t_{i-1} to t_i is denoted $\sigma_{i-1,i}$ and the unconditional volatility is $\sigma_{0,i}$.

To approximate the underlying asset process in equation (4.1.1) with a binomial process at time t_i, $i = 1, \cdots, m$, given the means μ_i, conditional volatilities $\sigma_{i-1,i}$, and the unconditional volatilities $\sigma_{0,i}$, HSS denote the conditional volatilities of the approximated binomial process as $\hat{\sigma}_{i-1,i}(n_i)$, where n_i denotes the number of binomial stages between time t_{i-1} and t_i, and they require that

$$\lim_{n_i \to \infty} \hat{\sigma}_{i-1,i}(n_i) = \sigma_{i-1,i}, \quad \forall i \quad (4.1.2)$$

It is similar to both the approximated unconditional volatility $\hat{\sigma}_{0,i}(n_1, n_2, \cdots, n_i)$ and the approximated mean $\hat{\mu}_i$ of the approximated binomial process, which require that

$$\lim_{n_l \to \infty} \hat{\sigma}_{0,i}(n_1, n_2, \cdots, n_i) = \sigma_{0,i}, \quad \forall i, l, \quad l = 1, \cdots, i \tag{4.1.3}$$

$$\lim_{n_i \to \infty} \hat{\mu}_i = \mu_i \tag{4.1.4}$$

Their method involves the construction of m separate binomial distribution, where the time periods are denoted $t_1, \cdots, t_i, \cdots, t_m$, and have the set of a discrete stochastic for X_i, where X_i is only defined at time t_i. In general they have the form of X_i at node *r*:

$$X_{i,r} = X_0 u_i^{N_i - r} d_i^r \tag{4.1.5}$$

where $N_i = \sum_{l=1}^{i} n_l$, and they have to determine the up and down movements u_i, d_i and the branching probabilities that satisfy the equations (4.1.2), (4.1.3) and (4.1.4). They denote

$$x_i = \ln(X_i / X_0)$$

and the probabilities to reach x_i given a node $x_{i-1,r}$ at t_{i-1} as

$$q(x_i \mid x_{i-1} = x_{i-1,r}) \text{ or } q(x_i)$$

An example, where $m = 2$ and we have X_0, X_1 and X_2 is illustrated in **Figure 1**.

Lemma 1 Suppose that the up and down movements u_i and d_i are chosen so that

$$d_i = \frac{2(E(X_i)/X_0)^{\frac{1}{N_i}}}{1 + \exp(2\sigma_{i-1,i}\sqrt{(t_i - t_{i-1})/n_i})}, \quad i = 1, 2, \cdots, m, \tag{4.1.6}$$

$$u_i = 2(E(X_i)/X_0)^{\frac{1}{N_i}} - d_i, \quad i = 1, 2, \cdots, m, \tag{4.1.7}$$

where $N_i = \sum_{l=1}^{i} n_l$, then if , for all *i,* the conditional probability $q(x_l) \to 0.5$ as $n_l \to \infty$, for $l = 1, \cdots, i$, then the unconditional mean and the conditional volatility of the approximated process approach respectively their true values:

$$\lim_{\substack{n_l \to \infty \\ l=1,\cdots,i}} \frac{\hat{E}(X_i)}{X_0} \to \frac{E(X_i)}{X_0}, \quad \lim_{n_i \to \infty} \hat{\sigma}_{i-1,i} \to \sigma_{i-1,i}$$

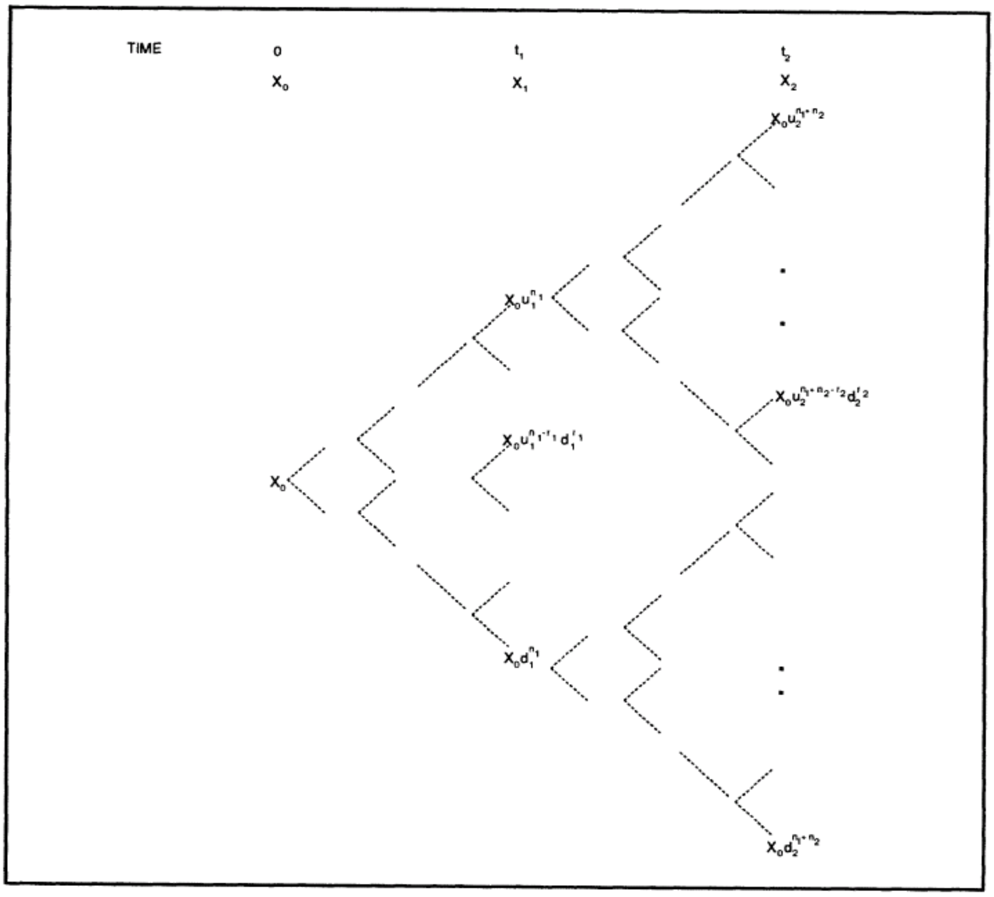

Figure 1 A discrete process for *X*$_1$, *X*$_2$

There are n_1+1 nodes at t_1 numbered r = 0, 1,⋯, n_1. There are n_1+ n_2+1 nodes at t_2 numbered r = 0, 1,⋯, n_{1+} n_2. X_0 is the starting price, X_1 is the price at time t_1, and X_2 is the price at time t_2. u_1, d_1, u_2 and d_2 are the proportionate up and down movements.

Since $x_i = \ln(X_i / X_0)$ is normally distributed, it follows that the regression

$$x_i = a_i + b_i x_{i-1} + \varepsilon_i, \quad E_{i-1}(\varepsilon_i) = 0$$

is linear with

$$b_i = \sqrt{[t_i \sigma_{0,i}^2 - (t_i - t_{i-1}) \sigma_{i-1,i}^2] / t_{i-1} \sigma_{0,i-1}^2} \ ,$$

and

$$a_i = E(x_i) - b_i E(x_{i-1})$$

They determined the conditional probabilities $q(x_i)$ so that

$$E_{i-1}(x_i) = a_i + b_i x_{i-1,r}$$

held for the approximated variables x_i and x_{i-1}.

Theorem 1 Suppose that the X_i are joint lognormally distributed. If the X_i are approximated with binomial distributions with $N_i = N_{i-1} + n_i$ stages and u_i and d_i given by equations (4.1.6) and (4.1.7), and if the conditional probability of an up movement at node *r* at time $i-1$ is

$$q(x_i \mid x_{i-1} = x_{i-1,r}) = \frac{a_i + b_i x_{i-1,r} - (N_{i-1} - r)\ln u_i - r\ln d_i}{n_i(\ln u_i - \ln d_i)} - \frac{\ln d_i}{\ln u_i - \ln d_i}, \quad \forall i,\ r \tag{4.1.8}$$

then $\hat{\mu}_i \to \mu_i$ and $\hat{\sigma}_{0,i} \to \sigma_{0,i}$ and $\hat{\sigma}_{i-1,i} \to \sigma_{i-1,i}$ as $n_i \to \infty,\ \forall i$

4.2 Applying the HSS Methodology to the LMM

After introducing the HSS methodology, we now apply this methodology into the LMM and make some change to satisfy our conventions. We have the following propositions.

Proposition 1 For the forward LIBOR rate which follows the lognormal distribution, we can choose the proper up and down movements to determine the i-th period of the T_n -maturity forward LIBOR rate and have the form

$$f(i;T_n,T_{n+1})_r = f(0;T_n,T_{n+1})u_i^{N_i - r}d_i^r,\ i = T_1, T_2, \cdots, T_n \tag{4.2.1}$$

where

$$d_i = \frac{2[E(f(i;T_n,T_{n+1}))/f(0;T_n,T_{n+1})]^{\frac{1}{N_i}}}{1+\exp(2\sigma_{i-1,i}\sqrt{(T_i - T_{i-1})/n_i})} \tag{4.2.2}$$

$$u_i = 2[E(f(i;T_n,T_{n+1}))/f(0;T_n,T_{n+1})] - d_i \tag{4.2.3}$$

$$N_i = N_{i-1} + n_i \tag{4.2.4}$$

r: node's number from top to bottom at time T_i

The structure of the binomial tree can be shown as **Figure 2**, with $n_1 + 1$ nodes at T_1 numbered $r = 0, 1, \dots, n_1$. There are nodes at T_2 numbered $r = 0, 1, \dots, n_1 + n_2$. Here we write the forward rate $f(0;2,3)$ in abbreviated form $f(0;2)$ and take $n_1 = n_2 = 2$, $r_1 = r_2 = 2$.

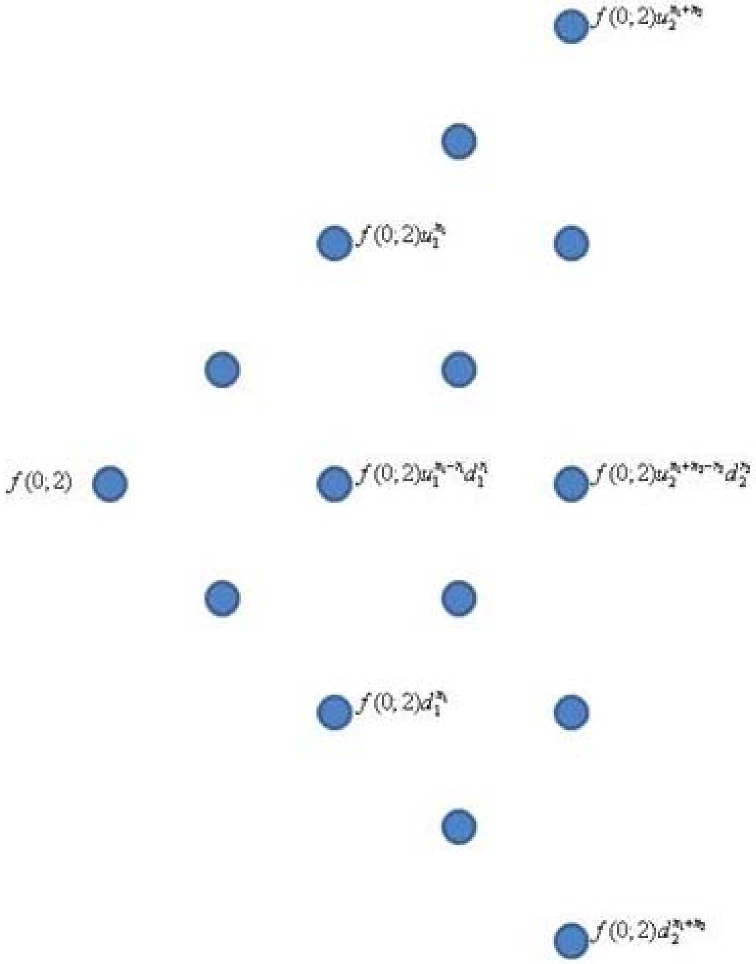

Figure 2 The binomial tree for the forward rate $f(0;2,3)$

After determining the structure of the forward LIBOR tree, we then have to choose the probability to satisfy the **Proposition 1**.

Proposition 2 Suppose that the forward LIBOR rate $f(i;T_n,T_{n+1})$ are joint lognormally distributed. If the $f(i;T_n,T_{n+1})$, $i=T_1,T_2,\cdots,T_n$ are approximated with binomial distributions with $N_i=N_{i-1}+n_i$ stages and u_i and d_i given by equations (4.2.2) and (4.2.3), and if the conditional probability of an up movement at node *r* at time T_i is

$$q(x_i \mid x_{i-1}=x_{i-1,r})=\frac{E_{i-1}(x_i)-(N_{i-1}-r)\ln u_i - r\ln d_i}{n_i(\ln u_i-\ln d_i)}-\frac{\ln d_i}{\ln u_i-\ln d_i} \quad \forall\ i,\ r \tag{4.2.5}$$

where

$$x_i=\ln\frac{f(i;T_n,T_{n+1})}{f(0;T_n,T_{n+1})} \tag{4.2.6}$$

$$E_{i-1}(x_i)=a_i+b_i x_{i-1,r}=E(x_i)-b_i E(x_{i-1})+b_i x_{i-1,r} \tag{4.2.7}$$

For determining the conditional probability, it has some skills to use for the term of $E_{i-1}(x_i)$ and following are the procedures to derive $E_{i-1}(x_i)$. We first derive $E(x_i)$ term in equation (4.2.7). Since the forward rate $f(i;T_n,T_{n+1})$ is lognormally distributed, we have

$$E(x_i)=\ln[\frac{E(f(i;T_n,T_{n+1}))}{f(0;T_n,T_{n+1})}]-\frac{1}{2}\sigma_{0,i}^2 \tag{4.2.8}$$

Second, we use the result of equation (3.2.7) obtained from the last section, and rewrite it as follows:

$$\begin{aligned}\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})}=1&+\frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n}+\frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n}\\&+\cdots+\frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n}\end{aligned} \tag{4.2.9}$$

Then multiple the $f(t;T_n,T_{n+1})/f(0;T_n,T_{n+1})$ term on both side to get the general form of $E_t(f(T_1;T_n,T_{n+1}))/f(0;T_n,T_{n+1})$:

$$\frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})}=$$
$$\frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times(1+\frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n}+\frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n}+\cdots+\frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n})$$

(4.2.9)

Finally, we substitute it into the formula (4.2.8) to obtain $E(x_i)$ term. Then, we take the value of $E(x_i)$ into equation (4.2.7) to compute the up movement probability at time T_i given the node $f(i-1;T_n,T_{n+1})_r$.

Note that when n_l stages approach the infinite $l=1,\cdots,i$, the sum of n_l stages also approach the infinite (i.e. $N_i=\sum_{l=1}^{i} n_l\to\infty$). We can reduce the up and down movements to the briefer form which is easier to calculate. That is

$$d_i=\frac{2}{1+\exp(2\sigma_{i-1,i}\sqrt{(T_i-T_{i-1})/n_i})}$$

$$u_i=2-d_i$$

and the conditional probability $q(x_l)\to 0.5$ as $n_l\to\infty$, for $l=1,\cdots,i$.

5 The Pricing of the Interest Rate Derivatives in the LMM

After we construct the forward tree process, it can be employed to price the derivatives. By beginning from the bond option on zero coupon bond (ZCB), then extend to the caplets.

5.1 The valuation of bond option on zero coupon bond in LMM

The bond option on ZCB is a bond that can be callable before maturity date with a callable price *K*. For example, we have a three years maturity zero coupon bond with a callable value *K* equal to 0.952381 dollar at year two. That is to say we can redeem the ZCB at year two with 0.952381 dollar or hold it until maturity at year three with 1 dollar. Therefore, we have to price the option value C_0 of this callable bond at time 0 (see the **Figure 3**)

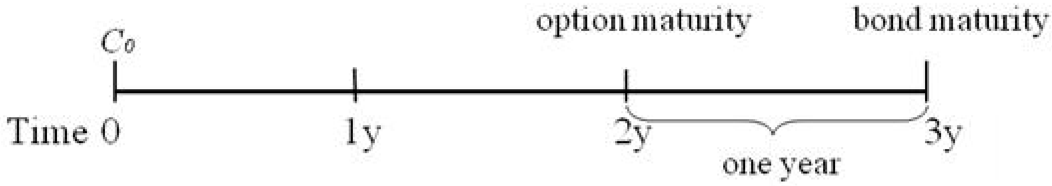

Figure 3 The callable bond for the 3 year maturity ZCB

To obtain the callable bond option value, we use the lattice method to price the option value of the callable bond. To get the payoff function at year two, we need know the zero coupon price at year two maturity at year three (i.e. *P*(2,3)). Comparing to the callable value *K*, we take function max(*P*(2,3)-K,0). Then we discount it back to the time 0 to get the option value of the callable bond. Here we take the flat forward rate 5% and constant volatility 10% and have

$$\begin{aligned} C_0 &= P(0,2) \times E[\max(P(2,3)-K,0)] \\ &= 0.90702948 \times 0.00258128 \\ &= 0.00234130 \end{aligned}$$

After having the option value at 10%volatility, we increase the volatility until reach 30% to see the relationship between option value and volatility. We plot the results into **Figure 4** to see the trend between option values and volatility.

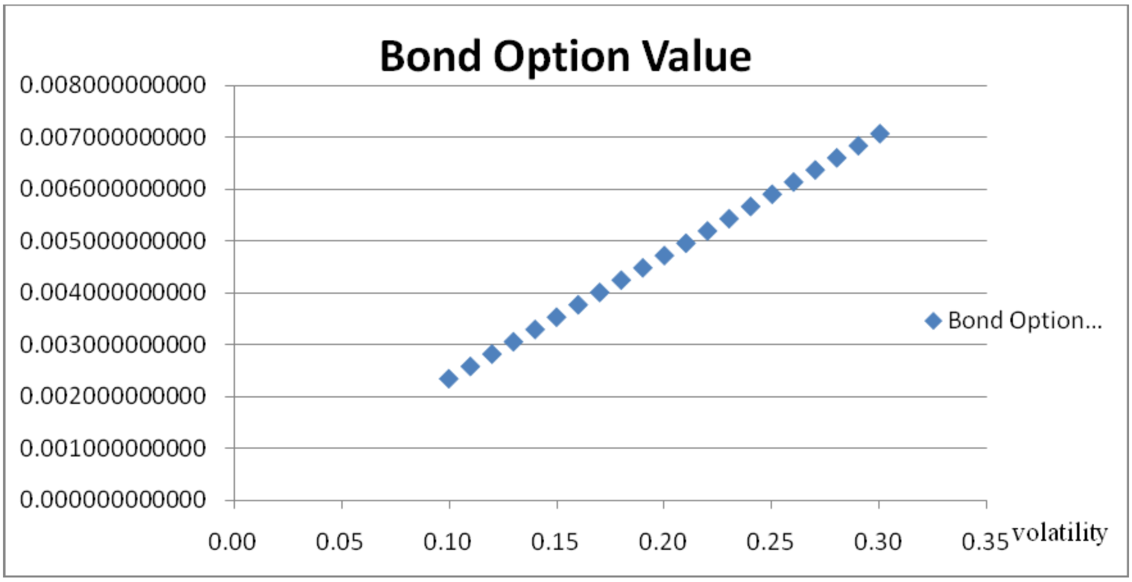

Figure 4 Bond option values for different volatility

From the above figure, we find that when the volatility increases, the value of bond option on ZCB increases. It is consistent with the inference for the Greek letter *vega* when the underlying asset's volatility increases the option value increases, too.

5.2 The valuation of caplets in LMM

A popular fixed income security is an interest rate cap, a contract that pays the difference between a variable interest rate applied to a principal and a fixed interest rate (strike price) applied to the same principal whenever the variable interest rate exceeds the fixed rate. We consider a cap with total life of T and let the tenor δ, the notional value A and the strike price K be fixed positive values. Note that the reset dates are T_1, T_2, … , T_n and define $T_{n+1} = T$. Define the forward rate $f(T_i;T_i,T_{i+1})$ as the future spot interest rate for the period between T_i and T_{i+1} observed at time T_i $(1 \le i \le n)$. The payoff function of a caplet at time T_{i+1} is

$$A \times \delta \times \max(f(T_i;T_i,T_{i+1}) - K, 0) \qquad (5.2.1)$$

Equation (5.2.1) is a caplet on the spot rate observed at time T_i with payoff occurring at time T_{i+1}. The cap is a portfolio consisted of n such call options which the underlying is known as caplet.

To derive out the price of the cap, we have to price the caplet first and then sum up the n caplets value to get the price of a cap. For a caplet price at time *t*, we use the Black's formula mentioned in chapter 3 (Equation (3.1.4)) to get the theoretical value. Thus, we restate the as follows:

$$caplet_i(t) = A \times \delta_i \times P(t, T_{i+1})[f(t; T_i, T_{i+1})N(d_1) - KN(d_2)] \tag{5.2.2}$$

where

$$d_1 = \frac{\ln(f(t; T_i, T_{i+1})/K) + \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$
$$d_2 = \frac{\ln(f(t; T_i, T_{i+1})/K) - \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$

After having the theoretical value as our benchmark, we use the payoff function to compute the price in the lattice method. To get the payoff function at time T_{i+1}, we have to know the evolution of the forward rate $f(0; T_i, T_{i+1})$ at time T_i. We construct the binomial tree of $f(0; T_i, T_{i+1})$ and known the $(f(T_i; T_i, T_{i+1})_r - K)^+$, r = 0, 1, …, i. Calculating the expectation of the payoff at time T_{i+1}, and then multiple the ZCB of $P(t, T_{i+1})$ to get the caplet value at time *t*.

In the followings, we take the 10 maturity of cap to compute the individual caplet from 1 period to 10 periods with the assumption of tenor δ and notional value *A* are equal to one and the volatility is constant and equal to 10%. Here the strike price *K* is 5%, the forward curve is flat 5% and the stages n_i for every period are equal to 25. We calculate one period caplet at time 0 ($caplet_1(0)$).

$$\begin{aligned} caplet_1(0) &= A\times\delta\times P(0,2)\times E[\max(f(1;1,2)-K,0)] \\ &= 1\cdot 1\cdot P(0,2)\cdot 0.0020112666 \\ &= 0.90702948\cdot 0.0020112666 \\ &= 0.0018242781 \end{aligned}$$

Table 1 Volatility is 10% and Stage n_i for Every Period is 25

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.0018085085	0.0018242781	0.0000157696	0.8719669656
2	0.0024348117	0.0024407546	0.0000059429	0.2440786919
3	0.0028388399	0.0028374958	-0.0000013441	0.0473484800
4	0.0031206153	0.0031282191	0.0000076038	0.2436631792
5	0.0033214311	0.0033204098	-0.0000010214	0.0307505629
6	0.0034637453	0.0034664184	0.0000026731	0.0771737036
7	0.0035616356	0.0035658574	0.0000042219	0.1185369137
8	0.0036247299	0.0036200240	-0.0000047059	0.1298286011
9	0.0036600091	0.0036633313	0.0000033221	0.0907678678
10	0.0036727489	0.0036743568	0.0000016079	0.0437804213
		RMSE	0.0000063671	

1. Caplet assume δ = 1 and stage 25
2. Assume volatility is 10%, the forward curve is flat 5%

Table 1 is the results for different maturity caplets. Besides the relative difference, we also use the RMSE to see the difference between the lattice value and Black's model for the whole maturity. The definition of the RMSE is given as follows:

RMSE (Root Mean Square Error)

A frequently-used measure of the differences between values predicted by a model or an estimator and the values actually observed from the thing being modeled or estimated. For the comparing difference between two models, the formula of RMSE can be expressed as

$$RMSE(\theta_1,\theta_2)=\sqrt{MSE(\theta_1,\theta_2)}=\sqrt{E((\theta_1-\theta_2)^2)}=\sqrt{\frac{\sum_{i=1}^{n}(x_{1,i}-x_{2,i})^2}{n}}$$

where

$$\theta_1 = \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,n} \end{bmatrix} \quad \text{and} \quad \theta_2 = \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ \vdots \\ x_{2,n} \end{bmatrix}$$

Here, θ_1 and θ_2 represent the lattice value and Black's model respectively which maturity form one to ten.

Now we change the stages from 25 to 50 to figure out the relationship between stages and RMSE. The results are shown in **Table 2**.

Table 2 Volatility is 10% and Stage n_i for Every Period is 50

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.0018085085	0.0018099405	0.0000014320	0.0791823392
2	0.0024348117	0.0024397802	0.0000049685	0.2040608652
3	0.0028388399	0.0028434461	0.0000046061	0.1622542164
4	0.0031206153	0.0031230795	0.0000024643	0.0789673074
5	0.0033214311	0.0033207434	-0.0000006878	0.0207066243
6	0.0034637453	0.0034620815	-0.0000016638	0.0480341136
7	0.0035616356	0.0035626664	0.0000010308	0.0289416315
8	0.0036247299	0.0036267433	0.0000020134	0.0555455781
9	0.0036600091	0.0036617883	0.0000017792	0.0486107386
10	0.0036727489	0.0036734197	0.0000006708	0.0182649876
		RMSE	0.0000025690	

1. Caplet assume δ = 1 and stage 50
2. Assume volatility is 10%, the forward curve is flat 5%

We also plot the RMSE with different stages between periods from 25 to 50 to see the convergence behavior of RMSE. **Figure 5** shows that the convergence behavior of RMSE for the different stages. We find that when we increase stages between periods, both relative difference and RMSE decrease and RMSE converge to zero with the stages go to infinite.

To see the impact of volatility on the value of different caplets and the convergence behavior of RMSE, we change the volatility from 10% to 20%. We do the same procedures as we do in volatility 10%, and results for the 25 and 50 stages are presented in **Table 3** and **Table 4** respectively. Finally, we plot the RMSE with different stages from 25 to 50 for volatility 20% in **Figure 6**.

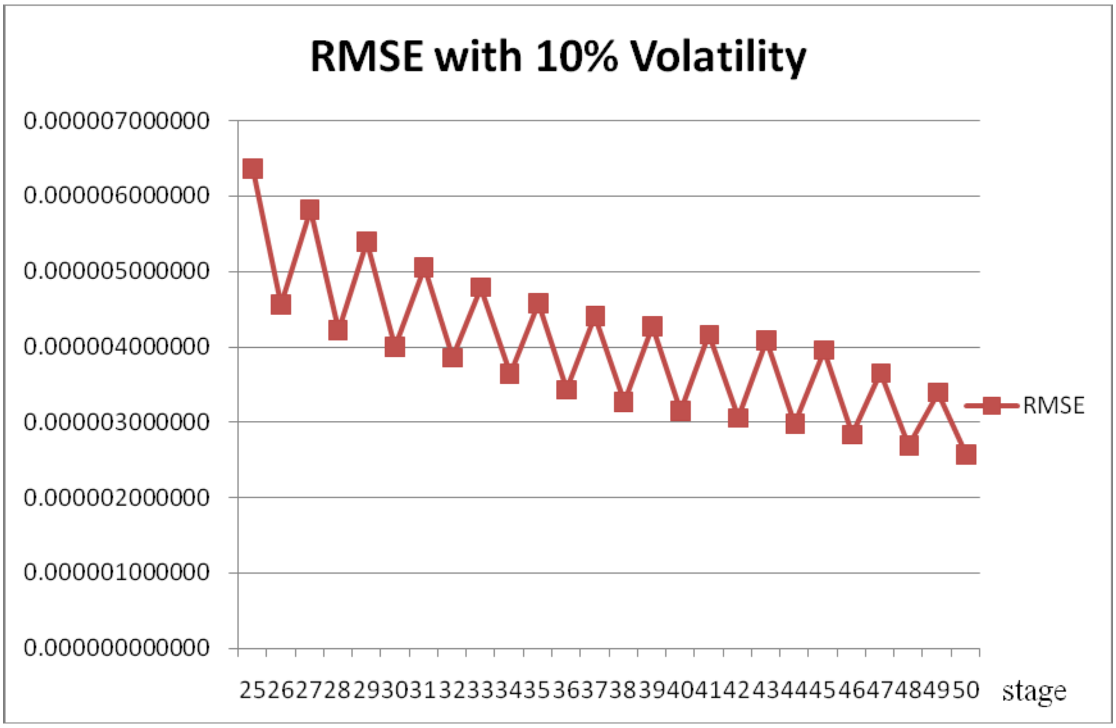

Figure 5 RMSE with Volatility 10%

Table 3 Volatility is 20% and Stage n_i for Every Period is 25

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.003612502	0.003629702	0.0000171997	0.4761162723
2	0.004857485	0.004880545	0.0000230598	0.4747275803
3	0.005656481	0.005664503	0.0000080217	0.1418144294
4	0.006210206	0.006193504	-0.0000167019	0.2689425754
5	0.006601645	0.006606642	0.0000049969	0.0756919625
6	0.006875986	0.006885656	0.0000096703	0.1406382089
7	0.007061574	0.007064876	0.0000033018	0.0467579666
8	0.007177804	0.007167478	-0.0000103258	0.1438570619
9	0.007238738	0.007241022	0.0000022831	0.0315395992
10	0.007255004	0.007260387	0.0000053836	0.0742052974
		RMSE	0.0000120045	

1. Caplet assume δ = 1 and stage **25**

2. Assume volatility is 20%, the forward curve is flat 5%

Table 4 Volatility is 20% and Stage n_i for Every Period is 50

Maturity	Black	Lattice	Difference	Relative Difference (%)
1	0.0036125022	0.0036270258	0.0000145236	0.4020371797
2	0.0048574848	0.0048648548	0.0000073700	0.1517253656
3	0.0056564814	0.0056507415	-0.0000057399	0.1014746318
4	0.0062102056	0.0062167021	0.0000064965	0.1046097077
5	0.0066016455	0.0066036633	0.0000020178	0.0305658301
6	0.0068759861	0.0068743634	-0.0000016227	0.0235991644
7	0.0070615740	0.0070656730	0.0000040990	0.0580467885
8	0.0071778037	0.0071775957	-0.0000002080	0.0028984108
9	0.0072387385	0.0072386556	-0.0000000829	0.0011446687
10	0.0072550037	0.0072576598	0.0000026562	0.0366117075
		RMSE	0.0000060911	

1. Caplet assume δ = 1 and stage **50**

2. Assume volatility is 20%, the forward curve is flat 5%

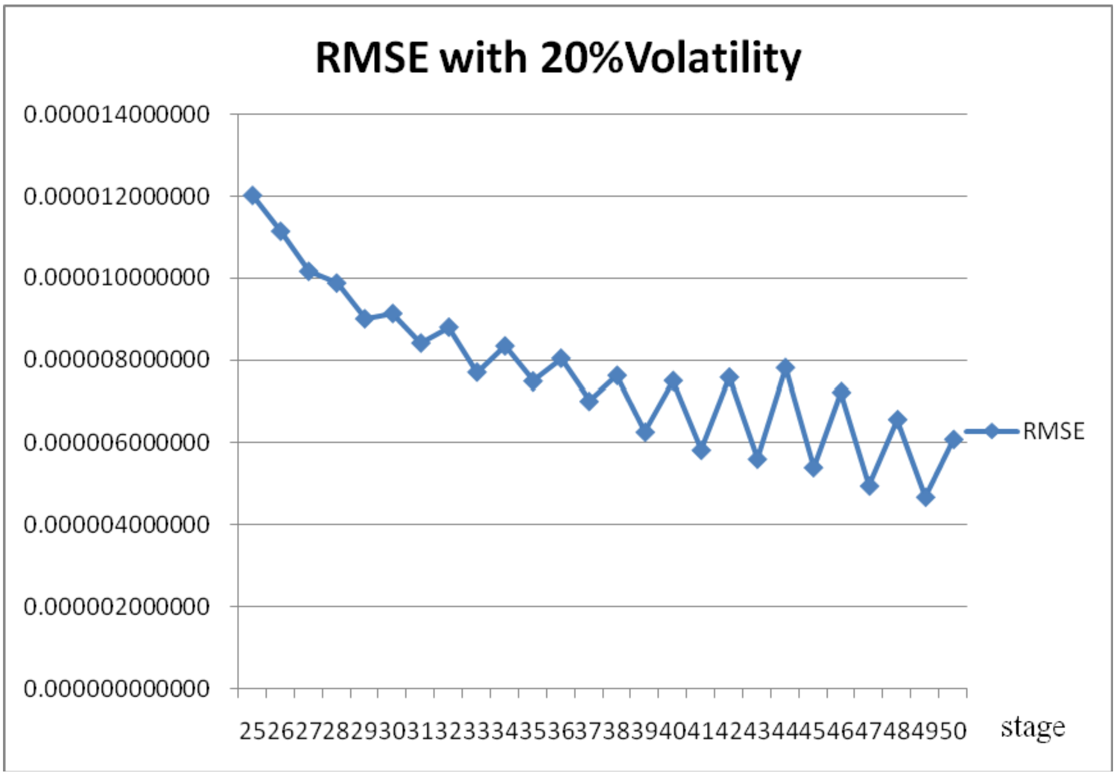

Figure 6 RMSE with Volatility 20%

We find that with the volatility increases, the value of caplets increases, and the convergence rate of RMSE decreases. It is consistent with high volatility makes the option value more valuable and convergence rate slower.

6 Conclusions

Implementing the LIBOR market model with lattice method is difficult. To make the pricing procedure by lattice method feasible, we construct a recombining binomial tree to depict the evolution of the forward LIBOR rate. In this model, we have all the forward rates for the different maturity at any node of the recombining binomial tree. With these rates on the nodes, we can easily figure out the early exercise decision for the American style derivatives which is a tough work in the Monte Carlo simulation.

After constructing the recombining binomial tree, the payoff of the interest rate derivatives on each node can be obtained. The pricing value of the derivatives can be calculated by backward induction method. We use the proposed model to calculate the value of bond option on zero coupon bond and caplets. Comparing to the theoretical value, we find the theoretical value and lattice method is close. However, with the stage between period by period increases, the difference between theoretical value and lattice method decreases. Besides, as the volatility increases the converge rate of RMSE decrease.

In the future, we have to find out the joint probability between different maturity forward rates and adjust the stages between period by period to fit the strike price to reduce the nonlinearity error. Trying to change the constant volatility to stochastic volatility to fit the volatility term structure will make the model more complete.

References

Brace, A., D. Gatarek, and M. Musiela, 1997, "The Market Model of Interest Rate Dynamics." *Mathematical Finance*, 7, no. 2, 127-155

Brigo D., F. Mercurio, 2006, *Interest Rate Models - Theory and Practice with Smile, Inflation and Credit*, 2^{nd} Edition, Springer.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53, 385-407

Derrick, S., D. Stapleton and R. Stapleton, 2005, "The Libor Market Model: A Recombining Binomial Tree Methodology." Working paper

Health, D., R. A. Jarrow, and A. Morton, 1990, "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." *Journal of Financial and Quantitative Analysis*, 25, no. 4, 419-440

Health, D., R. A. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology." *Econometrica*, 60, no. 1, 77-105

Ho, T. S. Y., and S.-B. Lee, 1986, "Term Structure Movements and Pricing Interest Rate Contingent Claims." *Journal of Finance*, 41, 1011-1029

Ho, T.S., R.C. Stapleton, and M.G. Subrahmanyam, 1995, "Multivariate Binomial Approximations for Asset Prices with Non-Stationary Variance and Covariance Characteristics," *Review of Financial Studies*, 8, 1125-1152.

Hull, J., 2006, *Options, Futures and Other Derivative Securities*, 6^{th} Edition, Pearson.

Hull, J. and A. White, 1990, "Pricing Interest Rate Derivative Securities." *Review of Financial Studies*, 3, no. 4, 573-592

Hull, J. and A. White, 1993, "Bond Option Pricing Based on a Model for the Evolution of Bond Prices." *Advances in Futures and Options Research*, 6, 1-13

Hull, J. and A. White, 1993, "The Pricing of Options on Interest Rate Caps and Floors Using the Hull-White Model." *Journal of Financial Engineering*, 2, no. 3, 287-296

Hull, J. and A. White, 1994, "Numerical Procedures for Implementing Term Structure Model I：Single Factor Models." *Journal of Derivatives*, 2, 7-16

Hull, J. and A. White, 1996, "Using Hull-White Interest Rate Trees." *Journal of Derivatives*, 26-36

Hull, J. and A. White, 2000, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model." *Journal of Fixed Income*, 10, no. 2, 46-62

Jamshidian, F., 1989, "An Exact Bond Option Formula." *Journal of Finance,* 44, 205-209

Jamshidian, F., 1997, "LIBOR and Swap Market Models and Measures." *Finance and Stochastics*, 1, 293-330

Miltersen, K., K. Sandmann, and D. Sondermann, 1997, "Closed Form Solutions for Term Structure Derivatives with Lognormal Interest Rates." *Journal of Finance*, 52, no. 1, 409-430

Nawalkha, S., N. A. Beliaeva and G. M. Soto, 2007, *Dynamic Term Structure Modeling*, Wiley.

Poon, S.H. and R.C. Stapleton, 2005, *Asset Pricing in Discrete Time A Complete Markets Approach*, Oxford.

Rebonato R., 2002, *Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond*, Princeton University Press.

Vasicek, O. A., 1977, "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5, 177-188