Pricing Snowball Notes with Hull-White Model

以 Hull-White 短利模型評價雪球型債券

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大綱

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雪球型債券簡介

• General form for i-th quarter coupon is:
  \[ \text{Coupon}(i) = (\text{Coupon}(i - 1) + \text{Spread}(i) - \text{Floating rate}(i))^+ \]

• Inversing floating rate bond
• Coupons freeze at zero
• Path-depend coupons:
  The more coupon rate in the past, snowball rolls more and more bigger. Oppositely, the less coupon rate before, the snowball will “melt away”.
• Delayed payment
• Callable contract
雪球型債券簡介

- Si: spread rate at time t(i)
- Ci: the coupon determined at time t(i) and paid at t(i+1)
- r(i): the floating interest rate from time t(i) to time t(i+1)
- F: the face value paid at maturity date t(n).

By recursive formula, Ci without freeze at zero article is following

\[
C_i = \left( C_{i-1} + S_i - r_i \right)
\]

\[
= \left[ \left( C_{i-2} + S_{i-1} - r_{i-1} \right) + S_i - r_i \right]
\]

\[
= \left[ \left( C_{i-3} + S_{i-2} - r_{i-2} \right) + S_{i-1} + S_i - r_{i-1} - r_i \right]
\]

\[
= \left( C_0 + \sum_{j=1}^{i} S_j - \sum_{j=1}^{i} r_j \right)
\]
研究方法

• Christian Bender, Anastasia Kolodko, John Schoenmakers, 2005, *Iterating Snowballs and related path dependent callables in a multi-factor Libor model*

• Pricing snowball based on Libor market model with iterative method.

• Shortcoming: estimate too many parameters in Libor market model, algorithm is sophisticated.

• We use simple interest rate model, Hull-White short rate model and trinomial tree structure to solve complex snowball contract.
Hull-White Model

- \( dr(t) = [ \theta(t) - ar(t) ] dt + \sigma dW(t) \)
- Character: property of mean reversion, fit to today’s term structure by \( \theta(t) \)
- Construct Hull-White trinomial by two stages:
  1. Preliminary Tree
     \( dR^*(t) = -aR^*(t)dt + \sigma dW(t) \)
  2. Calibration with the real term structure
     \( \alpha(t) = r(t) - R^*(t) \)
Hull-White trinomial tree
評價雪球型債券

(未考慮freeze at zero):

\[ C_i = (C_{i-1} + S_i - r_i) \]

\[ = \left[ (C_{i-2} + S_{i-1} - r_{i-1}) + S_i - r_i \right] = \left[ (C_{i-3} + S_{i-2} - r_{i-2}) + S_{i-1} + S_i - r_{i-1} - r_i \right] = \ldots \]

\[ = \left( C_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} r_k \right) \]

由Hull-White樹

\[ r_k = \alpha_k + f_{k,i} \Delta R \]

\[ C_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} \alpha_k - f \Delta R \]
建立變數狀態

• Given a path \{ node (0, 0) -> node (1, 1) -> node (2, 0) -> node (3, 1) \}

Rate at node(1,1) = 1*\Delta R
Rate at node(2,0) = 0*\Delta R
Rate at node(3,1) = 1*\Delta R

the coupon at node (3, 1) is

\[ C_0 + \sum_{k=1}^{3} (S_k - \alpha_k) - (1 + 0 + 1)\Delta R = C_0 + \sum_{k=1}^{3} (S_k - \alpha_k) - 2\Delta R \]
考慮加入狀態變數後
計算複雜度分析

• 找出 $f$ 的上下限

$$\text{Upper } \leq 1 + 2 + \ldots + n - 1 = \frac{n(n - 1)}{2}$$
$$\text{Lower } \geq -1 - 2 - \ldots - (n - 1) = \frac{-n(n - 1)}{2}$$

• 每個節點最多需 $n(n-1)+1$ 個狀態變數

• 樹的節點個數上限 = $1 + 3 + \ldots + (2n + 1) = \frac{(n + 1)(2n + 2)}{2}$

$$\frac{(n + 1)(2n + 2)}{2} \times (n \times (n - 1) + 1) \rightarrow O(n^4)$$

電腦可處理
• The node \((i, j)\) in the Hull-White preliminary tree means that at time \(i\Delta t\), the rate is \(j\Delta R\), where \(\Delta R = \sigma \sqrt{3\Delta t}\).
Example

node A : $\text{Sum}(2, 2) = \{-3\}$
node B : $\text{Sum}(2, 1) = \{-1, -2\}$
node D : $\text{Sum}(2, 0) = \{1, 0, -1\}$

$Min(\text{Sum}(2, j^\prime)) - 1 \leq x \leq Max(\text{Sum}(2, j^\prime)) - 1$

where $x \in \text{Sum}(3, 1) j^\prime \in \{0, 1, 2\}$

$\Rightarrow Min(\text{Sum}(2, 2)) - 1 \leq x \leq Max(\text{Sum}(2, 0)) - 1$

$\Rightarrow -4 \leq x \leq 0$

$\therefore \text{Sum}(3, 1) = \{0, -1, -2, -3, -4\}$

$C_{3,1} = \{ \sum_{k=1}^{i} (S_k - \alpha_k) - 0 \Delta R, \sum_{k=1}^{i} (S_k - \alpha_k) - 1 \Delta R, \sum_{k=1}^{i} (S_k - \alpha_k) - 2 \Delta R, \sum_{k=1}^{i} (S_k - \alpha_k) - 3 \Delta R, \sum_{k=1}^{i} (S_k - \alpha_k) - 4 \Delta R \}$
Freeze at zero 限制

- 本期的债券利率 = 上期债券利率 + Inverse floater
  - 当利率 < 0 时 债券利率重设为 0
- 第i 期利率可能无法写成 $C_0 + \sum_{j=1}^{i} S_j - \sum_{j=1}^{i} \alpha_j - f \Delta R$
  - Ex:
    $C_1 + S_2 - r_2 < 0 \rightarrow C_2 = (C_1 + S_2 - r_2)^+ = 0$
    $C_3 = (S_3 - r_3)^+$

- 處理方法：
  - 拿掉 $C_0 + \sum_{j=1}^{i} S_j - \sum_{j=1}^{i} R_j - f \Delta R$ 中小于 0 的状态
  - 加入状态 0* 债券利率为 0 的Case
  - 使用内插计算 0* 所对应的価格
在 Freeze at zero 限制下，找出 $f$ 的上下限

- Suppose $C0=0\%$
- Step 1: 拿掉小於 0 的狀態，加入狀態 $0^*$

$$
\exists k_i \in \mathbb{Z} \ s.t. \ \sum_{k=1}^{i} (S_k - \alpha_k) + k_i \Delta R \geq 0
$$

$$
\Rightarrow k_i \geq \sum_{k=1}^{i} \frac{(S_k - \alpha_k)}{\Delta R}, \quad k_i = \left\lfloor \sum_{k=1}^{i} \frac{(S_k - \alpha_k)}{\Delta R} \right\rfloor
$$
**STEP 1**

Check each node \((i, j)\) at \(i=2\)

Let \(S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03\)

\(\Delta R = 0.0052\)

\(\alpha1 = 0.021319, \alpha2 = 0.02303\)

\(\alpha3 = 0.02397, \alpha4 = 0.0244336\)

\(S1 + S2 - \alpha1 - \alpha2 + k_2 \cdot \Delta R \geq 0 \quad \forall i = 2\)

\(\Rightarrow k_2 = \frac{\alpha1 + \alpha2 - S1 - S2}{\Delta R} = \frac{3.999807692}{0.0052} = -2.999807692\)

\(\therefore k_2 = \frac{\alpha1 + \alpha2 - S1 - S2}{\Delta R} = -2\)

\(\Rightarrow\text{Reset node}(2,2)\) to “0*”

\(\Rightarrow\text{Change} (-3,-3)\) to \((0*,0*)\)

When \(\sum(2,j) \geq -2\), coupon rate > 0

\(\Rightarrow\) No change other node except node(2,2)
Maximum and minimum

STEP 1

⇒ Reset node(2,2) to “0*”
⇒ Change (-3,-3) to (0*,0*)

If Sum(2,j) >= -2, coupon rate > 0

For example

$$\text{Coupon}_{2,1} = \left\{ \sum_{k=1}^{2} (S_k - \alpha_k) - \Delta R, \sum_{k=1}^{2} (S_k - \alpha_k) - 2\Delta R \right\}$$

$$\text{Coupon}_{2,2} = \{0\}$$
在Freeze at zero 限制下，找出f的上下限

- Suppose C0=0%
- Step 2: 計算前一個node有0*的狀態

$$\exists \delta_{i,j} \in \mathbb{Z} \ s.t. \ \sum_{k=1}^{i} (S_k - \alpha_k) + \delta_{i,j} \Delta R \approx z_{i,j}$$

where $$z_{i,j} = S_i - (\alpha_i + f_{i,j}) = (S_i - \alpha_i) - j\Delta R$$

$$\Rightarrow \delta_{i,j} = \left[ -(\sum_{k=1}^{i-1} (S_k - \alpha_k) + j\Delta R) \right] \frac{\Delta R}{\Delta R}$$

- 使重設的利率也可寫成$$\left( C_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} \alpha_k - f\Delta R \right)$$的形式
  取下高斯=>為了利用內插法計算債券價值
STEP 2

Construct each node \((i, j)\) at \(i=3\)

Let \(\delta(3,3)\) \( s.t. \)
\[ S3 - \alpha_3 - 3* \Delta R \]
\[ = S1 + S2 + S3 - \alpha_1 - \alpha_2 - \alpha_3 + \delta(3,3)* \Delta R \]
\[ \therefore \delta(3,3) = \left[ \frac{\alpha_1 + \alpha_2 - S1 - S2 - 3* \Delta R}{\Delta R} \right] = -7 \]

By the same way
\[ \delta(3,2) = \left[ \frac{\alpha_1 + \alpha_2 - S1 - S2 - 2* \Delta R}{\Delta R} \right] = -6 \]
\[ \delta(3,1) = \left[ \frac{\alpha_1 + \alpha_2 - S1 - S2 - 1* \Delta R}{\Delta R} \right] = -5 \]
STEP 3

Construct each node \((i, j)\) at \(i=3\)

let \(S_1 = 0.03, S_2 = 0.03, S_3 = 0.03, S_4 = 0.03\)

\[\Delta R = 0.0052\]

\[\alpha_1 = 0.021319, \alpha_2 = 0.02303\]

\[\alpha_3 = 0.02397, \alpha_4 = 0.0244336\]

\[S_1 + S_2 + S_3 - \alpha_1 - \alpha_2 - \alpha_3 + k_3 \Delta R > 0 \quad \forall N = 3\]

\[\Rightarrow k_3 > \frac{\alpha_1 + \alpha_2 + \alpha_3 - S_1 - S_2 - S_3}{\Delta R}\]

\[\therefore k_3 = \left[\frac{\alpha_1 + \alpha_2 + \alpha_3 - S_1 - S_2 - S_3}{\Delta R}\right] = -4\]

Reset node(3,3), node(3,2), and node(3,1).

Change \((-7,-7)\) to \((0^*,0^*)\)

\((-3,-6)\) to \((-3,-4,0^*)\)

\((0,-5)\) to \((0,-4,0^*)\)
STEP 4

Return to Step 2, construct the Maximum and minimum of next period, and reiterate Step 2,3,4 until final period of interest rate.

Note: 0* means that node have the state of resetting coupon rate and zero coupon rate

$$C_{3,2} = \{0, \sum_{k=1}^{i} (S_k - \alpha_k) - 3\Delta R, \sum_{k=1}^{i} (S_k - \alpha_k) - 4\Delta R\}$$
STEP 5

Save possible state of each node in the same period.

ex. (N=3)

<table>
<thead>
<tr>
<th>Node (N, j)</th>
<th>Maximum</th>
<th>Minimum</th>
<th>0*(1 or 0)</th>
<th>total</th>
</tr>
</thead>
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<td>(3, 3)</td>
<td>X</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>-3</td>
<td>-4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>1</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2, -2)</td>
<td>-4</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>0</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(0, -4)</td>
<td>-3</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>X</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Spacing between inverse interest rate (-j)

Period of interest rate

Maximum and minimum

(0*, 0*)

(-1, -1)

(0, 0)

(1, 1)

(2, -2)

(3, 3)

(-1, -2)

(0, -4)

(1, -1)

(5, 3)

(2, 1)

(4, 0)

(-3, -4, 0*)

(0*, 0*)

(0*, 0*)

(6, 6)
計算 snowball 債券價值

B (i, j, Sum(a)) means bond value at node (i, j) with Sum (i, j) = a.

Example 1:

\[
\begin{align*}
\text{Sum}(a_u) &= \text{Sum}(-4) - j_u \\
&= -4 - 2 = -6 < k_4 \\
\text{Sum}(a_m) &= \text{Sum}(-4) - j_m \\
&= -4 - 1 = -5 \\
\text{Sum}(a_d) &= \text{Sum}(-4) - j_d \\
&= -4 - 0 = -4
\end{align*}
\]

\[
\text{Sum}(a_u) = \text{Sum}(0^*) \\
\text{Sum}(a_m) = \text{Sum}(-5) \\
\text{Sum}(a_d) = \text{Sum}(-4)
\]

\[D(3,1,\text{Sum}(-4)) = (P_u B(4,2,\text{Sum}(0^*)) + P_m B(4,1,\text{Sum}(-5)) + P_d B(4,0,\text{Sum}(-4))) * \frac{1}{1 + (\alpha_3 + 1* \Delta R) * (t_4 - t_3)}
\]

\[B(3,1,\text{Sum}(-4)) = \{\min(D(3,1,\text{Sum}(-4)), 1) + C; \forall C \in \{C_{2,j^*}: \text{Sum}(2,j^*) - 1 = -4, j^* = 0,1,2\}\}
\]
Example 2

- Actual coupon rate $\chi_{i,j}$ fixed
  
  at node $(4, j')$, $j' = \{1, 2, 3\}$ is following:

  $$\chi_{i,j} = \max(S_i - (\alpha_i + f_{i,j} \Delta R), 0)$$

  let $S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03$
  
  $\Delta R = 0.0052$
  
  $\alpha_1 = 0.021319, \alpha_2 = 0.02325$
  
  $\alpha_3 = 0.02397, \alpha_4 = 0.0244336$
  
  $\chi_{4,3} = \max(S_4 - \alpha_4 - 3\Delta R, 0) = 0$
  
  $\chi_{4,2} = \max(S_4 - \alpha_4 - 2\Delta R, 0) = 0$
  
  $\chi_{4,1} = \max(S_4 - \alpha_4 - 1\Delta R, 0) = 0.0003664$

  Let $k^*$ s.t. $S4 - \alpha_4 - 1^* \Delta R = \sum_{k=1}^{4} (S_k - \alpha_k) + k^* \Delta R$

  $k^* = -5.169423077$

  $\therefore$ Actual coupon rate $\chi_{4,1} \in (C_{4,1, Sum(-5)}, C_{4,1, Sum(0*)})$
Actual discounted bond value from node(4,1) is calculated by interpolation method:

\[ C_{4,1,\text{Sum}(-5)} : \text{Actual coupon at node(4,1)} \]

\[ = \frac{[B(4,1,\text{Sum}(-5)) - B(4,1,\text{Sum}(0*))] : \text{[Actual bond value at node(4,1) - } B(4,1,\text{Sum}(0*))]} {C_{4,1,\text{Sum}(-5)}} \]

\[ = \text{Actual reset coupon at node(4,1)} \times \frac{[B(4,1,\text{Sum}(-5)) - B(4,1,\text{Sum}(0*))]} {C_{4,1,\text{Sum}(-5)}} + B(4,1,\text{Sum}(0*)) \]
用線性內插法求 snowball 價值 (3)

Summation in node(i,j)

\begin{align*}
&j = 2 \\
&\text{node}(0^*) \\
&\quad \rightarrow j = 3 \\
&\quad \rightarrow j = 2 \\
&\quad \rightarrow (-5, 0^*) \quad j = 1
\end{align*}

\begin{align*}
i = 3 & \quad \text{Summation in node}(i,j) \\
&\quad \rightarrow 0^* \\
&\quad \rightarrow 0^* \\
&\quad \rightarrow (-5, 0^*) \\
&\quad \rightarrow 0^* \\
&\quad \rightarrow 0^* \\
&\quad \rightarrow (-5, 0^*)
\end{align*}

\begin{align*}
D(3,2, Sum(0*)) &= (P_u B(4,3, Sum(0*))) + P_m B(4,2, Sum(0*)) \\
&\quad + P_d * \text{Actual bond value}(4,1)) * \frac{1}{1 + (\alpha_3 + 2^* \Delta R) * (t_4 - t_3)}
\end{align*}

\begin{align*}
B(3,2, Sum(0*)) &= \{ \min(D(3,2, Sum(0*)), 1) + C; \forall C \in \{C_{2,i} : Sum(2, j^*) - 2 = 0^*, j^* = 1, 2 \}\}
\end{align*}
敏感度分析

- A snowball note issued by Bank SinoPac which the contract could be redeemed with par value after the third year.
- C\textsubscript{n, i} : coupon rate at i quarter of n year.
- FR : the fixing rate of 90 days CP; if i-1=0, C\textsubscript{n, i-1}= C\textsubscript{n-1, 4} for n=1…10, i=1..4.

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<thead>
<tr>
<th>Year</th>
<th>Coupon rate (C\textsubscript{n, i})</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>C1,\textsubscript{i}=3%, i=1,2,3,4</td>
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<td>2</td>
<td>C2,\textsubscript{i}=C2,\textsubscript{i-1}+1.40%-FR\textsubscript{2, i}</td>
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<td>C4,\textsubscript{i}=C4,\textsubscript{i-1}+1.90%-FR\textsubscript{4, i}</td>
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<td>C5,\textsubscript{i}=C5,\textsubscript{i-1}+2.15%-FR\textsubscript{5, i}</td>
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<td>C10,\textsubscript{i}=C10,\textsubscript{i-1}+3.40%-FR\textsubscript{10, i}</td>
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</table>
Price vs. Volatility

\[ Coupon(i) = (Coupon(i - 1) + Spread(i) - Floating\ rate(i))^+ \]

Snowball Price vs. Volatility of Hull-White Model
敏感度分析(2)
Price vs. Mean Reversion

Snowball Price vs. Mean Reversion

- callable contract
- noncallable contract
Price vs. Market Rates

**Coupon(i)** = (**Coupon(i - 1)** + **Spread(i)** - **Floating rate(i))**

Snowball Notes with Mean Reversion 0.1 vs. Zero Curve
Redeemable Snowball Price vs. Spread Article

$\text{Coupon}(i) = (\text{Coupon}(i - 1) + \text{Spread}(i) - \text{Floating rate}(i))^+$

Snowball with Redemption Article vs. Spread
Face Value = $1

Twice spread  Half spread  One spread
Non-redeemable Snowball Price vs. Spread Article

\[ Coupon(i) = (Coupon(i - 1) + Spread(i) - Floating rate(i))^{+} \]

Snowball without Redemption Article vs. Spread

Face value = $1

[Graph showing the relationship between Price, Volatility of Hull-White Model, and different spread scenarios: Twice spread, Half spread, One spread, One spread with callable contract.]
敏感度分析(5)
Snowball Price vs. Redemptive Time

Snowball Price v.s. Redemptive Time

price v.s. callable_time

Year

Price

1 2 3 4 5 6 7 8 9 10
實證結果

- Zero rates data from CHINA BILLS FINANCE CORPORATION (中華票券)

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<td>2.4790%</td>
<td>2.4966%</td>
</tr>
</tbody>
</table>
• Mean reversion and volatility are constants in Hull-White model.
• Use cap price to calibrate parameters.
• A popular goodness-of-fit measure is

\[
SSE = \min_{a, \sigma} \sum_{k} \sum_{i=1}^{n} (U_{ik} - V_{ik})^2
\]

\[k \in \{1.5\%, 2.5\%, 3.5\%, 4.5\%\}\]

where \( U_i \) is the market price calculated by Black’s equation and \( V_i \) is the price given by the Hull-White model.
Cap is a portfolio of interest rate options of caplet

\( T_r \) : Total life of cap
\( K_{R\_cap} \) : Cap rate
\( \Delta k \) : Period time, \( \Delta k = t_{k+1} - t_k \)
\( L \) : Principal
\( r_k \) : Interest rate for period between time \( t_k \) and \( t_{k+1} \)
\( F_t \) : Forward rate for period between time \( t_k \) and \( t_{k+1} \)

\[ \text{Caplet} = L\Delta k \max(r_k - K_{R\_cap}, 0) \]

- Caplet present value by Black’s equation

\[ L\Delta k P(0, t_{k+1})[F_k N(d_1) - K_{R\_cap} N(d_2)] \]

\[ d_1 = \frac{\ln[F_k / K_{R\_cap}] + \sigma_k^2 t_k / 2}{\sigma_k \sqrt{t_k}} \]
\[ d_2 = d_1 - \sigma_k \sqrt{t_k} \]
Cap is a portfolio of bond options

- Interest rate cap could be characterized as a portfolio of put options on zero-coupon bond.
- Caplet as a put options on zero-coupon bond:

\[
(1 + K_{R\_cap} \Delta k) \max(K_{cap} - S, 0)
\]

\[
S = \frac{1}{1 + r_k \Delta k} \quad K_{cap} = \frac{1}{1 + K_{R\_cap} \Delta k}
\]

- put option by Hull-White model:

\[
KP(0, T)N(-h + \sigma_p) - LP(0, s)N(-h)
\]

\[
h = \frac{1}{\sigma_p} \ln \frac{LP(0, s)}{KP(0, T)} + \frac{\sigma_p}{2} \quad \sigma_p = \frac{\sigma}{a} \left[1-e^{-a(s-T)}\right] \sqrt{\frac{1-e^{-2aT}}{2a}}
\]
Estimate mean reversion and Hull-Whit vol. with different strike price (k)  
(a, sigma)= (0.014485, 0.004596)

The minimum summation of square error (SSE) = 7.80133e-005
### Snowball Notes

Par value = Market price = $1

<table>
<thead>
<tr>
<th>Year</th>
<th>Coupon rate (Cn,i)</th>
<th>Non-redeemable snowball price</th>
<th>Redeemable snowball price</th>
<th>Option value for redemptive article</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C1,i=3%, i=1,2,3,4</td>
<td></td>
<td></td>
<td>1.13901</td>
</tr>
<tr>
<td>2</td>
<td>C2,i=C2,i-1+1.40%-FR2,i</td>
<td></td>
<td></td>
<td>0.880214</td>
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<tr>
<td>3</td>
<td>C3,i=C3,i-1+1.65%-FR3,i</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>C4,i=C4,i-1+1.90%-FR4,i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>C5,i=C5,i-1+2.15%-FR5,i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>C6,i=C6,i-1+2.40%-FR6,i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>C7,i=C7,i-1+2.65%-FR7,i</td>
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<td></td>
</tr>
<tr>
<td>8</td>
<td>C8,i=C8,i-1+2.90%-FR8,i</td>
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<td></td>
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</tr>
<tr>
<td>9</td>
<td>C9,i=C9,i-1+3.15%-FR9,i</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>C10,i=C10,i-1+3.40%-FR10,i</td>
<td></td>
<td></td>
<td>0.258796</td>
</tr>
</tbody>
</table>
結論

• Snowball Notes is an inverse floating rate bond with freeze at zero and redemptive articles.

• Contribution: Provide an innovative polynomial-time pricing algorithm based on simple term structure model, Hull-White short rate model, to price snowball notes.

• Sensitivity analysis: Snowball price has negative relation to mean reversion and market rate, and positive relation to volatility, spreads and redemptive time.

• Issuers can effectively hedge by redemptive articles.
Thanks for your listening!