Pricing Snowball Notes with Hull-White Model

以 Hull-White 短利模型評價雪球型債券

研究生：戴　慈

指導教授：王克陸　博士
戴天時　博士

大　綱

- 雪球型債券簡介
- 研究方法
- Hull-White Model
- 評價雪球型債券
  - 建立變數狀態
  - Freeze at zero 限制
  - 計算snowball債券價值
- 敏感度分析
- 實證結果
- 結　　論

雪球型債券簡介

- General form for i-th quarter coupon is:

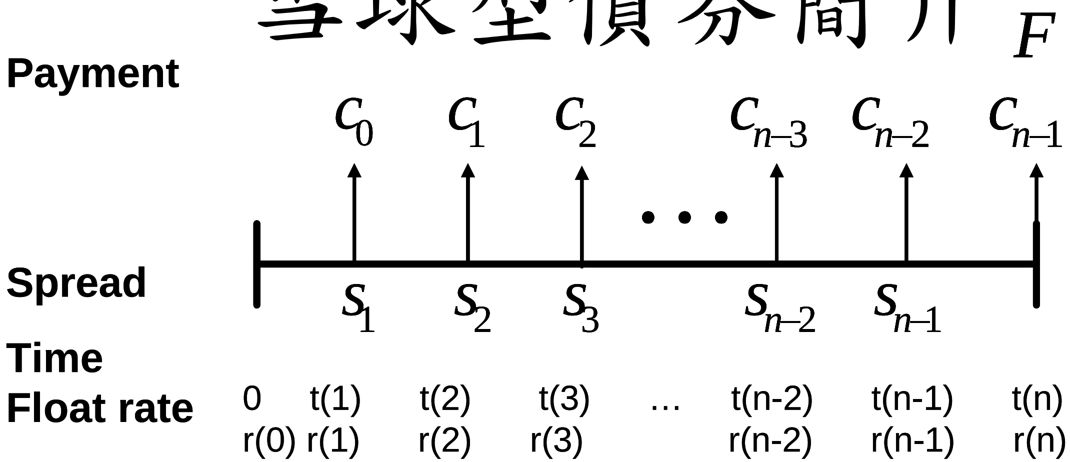
$$Coupon(i) = (Coupon(i-1) + Spread(i) - Floating\ rate(i))^{+}$$

- Inversing floating rate bond
- Coupons freeze at zero
- Path-depend coupons:

  The more coupon rate in the past, snowball rolls more and more bigger. Oppositely, the less coupon rate before, the snowball will “melt away”.

- Delayed payment
- Callable contract

雪球型債券簡介



- Si : spread rate at time t(i)
- Ci : the coupon determined at time t(i) and paid at t(i+1)
- r(i): the floating interest rate from time t(i) to time t(i+1)
- F: the face value paid at maturity date t(n).
- By recursive formula, Ci without freeze at zero article is following

$$
\begin{aligned}
C_i &= \left(C_{i-1} + S_i - r_i\right) \\
&= \left[\left(C_{i-2} + S_{i-1} - r_{i-1}\right) + S_i - r_i\right] \\
&= \left[\left(C_{i-3} + S_{i-2} - r_{i-2}\right) + S_{i-1} + S_i - r_{i-1} - r_i\right] \\
&= \left(C_0 + \sum_{j=1}^{i} S_j - \sum_{j=1}^{i} r_j\right)
\end{aligned}
$$

研究方法

- Christian Bender, Anastasia Kolodko, John Schoenmakers, 2005, *Iterating Snowballs and related path dependent callables in a multi-factor Libor model*
- Pricing snowball based on Libor market model with iterative method.
- Shortcoming: estimate too many parameters in Libor market model, algorithm is sophisticated.
- We use simple interest rate model, Hull-White short rate model and trinomial tree structure to solve complex snowball contract.

Hull-White Model

- $dr(\mathrm{t}) = [\,\theta(t) - ar(t)\,]dt + \sigma\, dW(t)$
- Character: property of mean reversion, fit to today's term structure by $\theta(t)$
- Construct Hull-White trinomial by two stages:

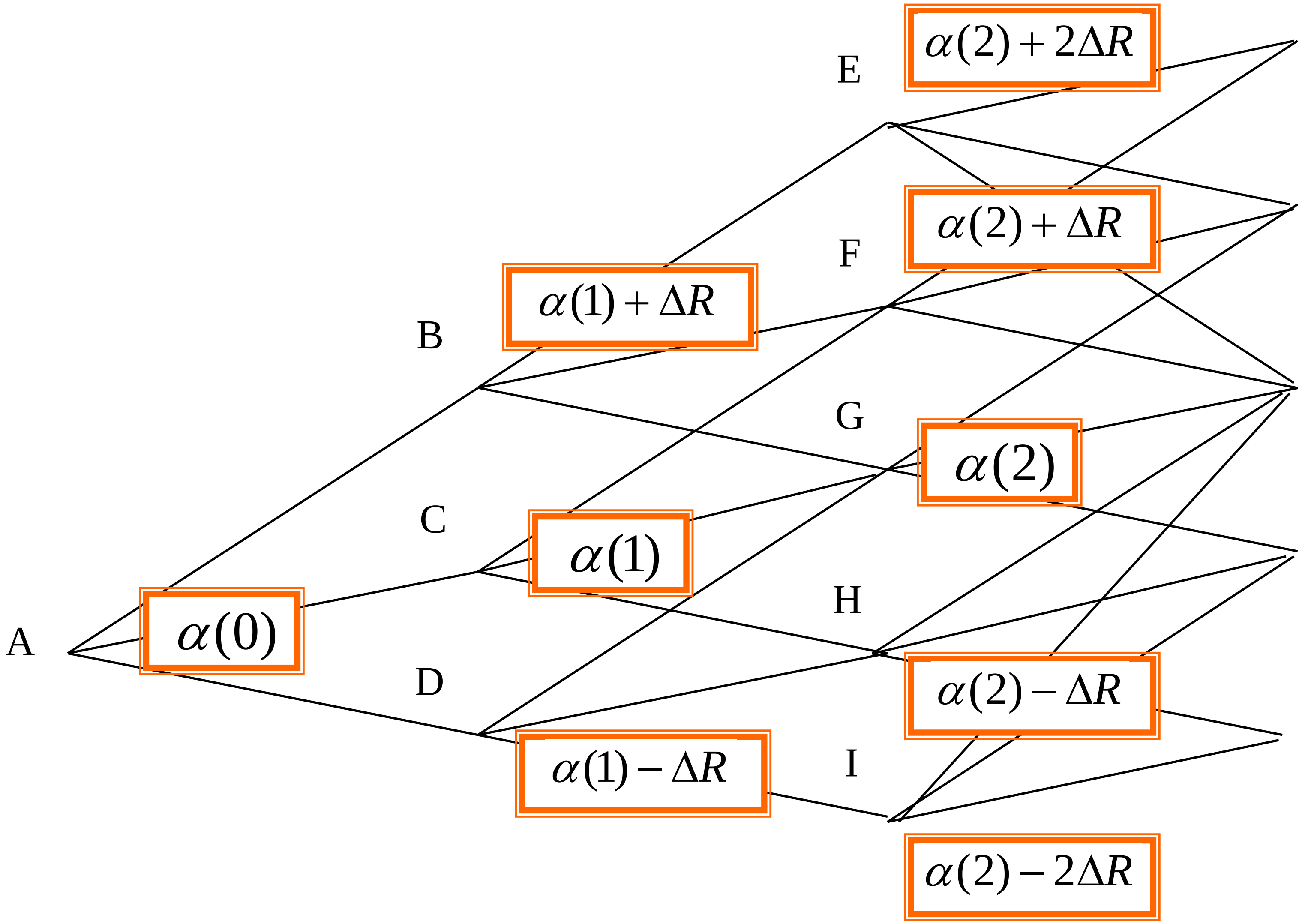
1. Preliminary Tree

$dR^*(\mathrm{t}) = -aR^*(t)dt + \sigma\, dW(t)$

2. calibration with the real term structure

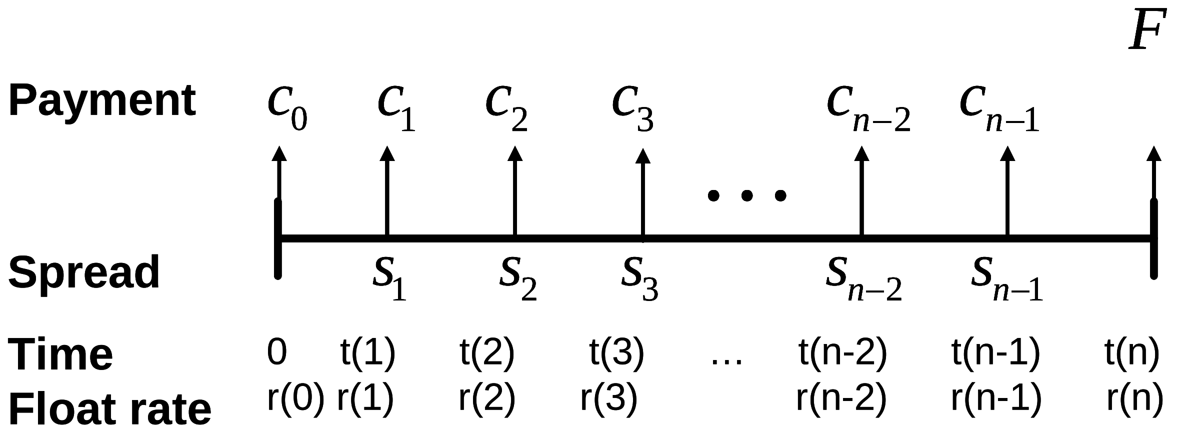
$\alpha(t) = r(t) - R^*(t)$

Hull-White trinomial tree



評價雪球型債券

- (未考慮freeze at zero):



$$C_i = \left(C_{i-1} + S_i - r_i\right)$$

$$= \left[\left(C_{i-2} + S_{i-1} - r_{i-1}\right) + S_i - r_i\right]$$

$$= \left[\left(C_{i-3} + S_{i-2} - r_{i-2}\right) + S_{i-1} + S_i - r_{i-1} - r_i\right] = \ldots$$

$$= \left(C_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} r_k\right) \xrightarrow[r_k = \alpha_k + f_{k,j_k}\Delta R]{\text{by Hull-White tree}} \left(C_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} \alpha_k - f\,\Delta R\right)$$

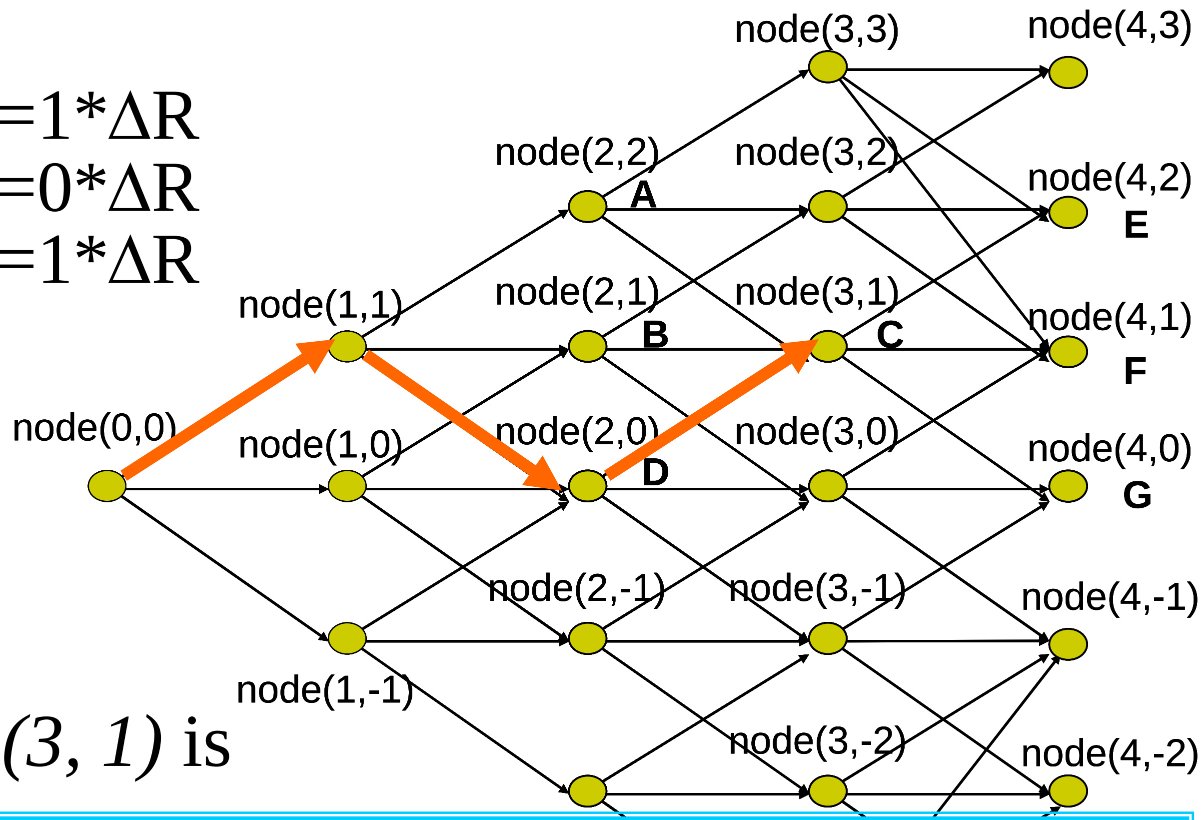
Integer

Constant

建立變數狀態

- Given a path { *node (0, 0)-> node (1, 1) -> node (2, 0) -> node (3, 1)*}

Rate at node(1,1) =1*ΔR
Rate at node(2,0) =0*ΔR
Rate at node(3,1) =1*ΔR



the coupon at *node (3, 1)* is

$$C_0 + \sum_{k=1}^{3}(S_k - \alpha_k) - (1+0+1)\Delta R = C_0 + \sum_{k=1}^{3}(S_k - \alpha_k) - 2\Delta R$$

考慮加入狀態變數後
計算複雜度分析

- 找出 f 的上下限

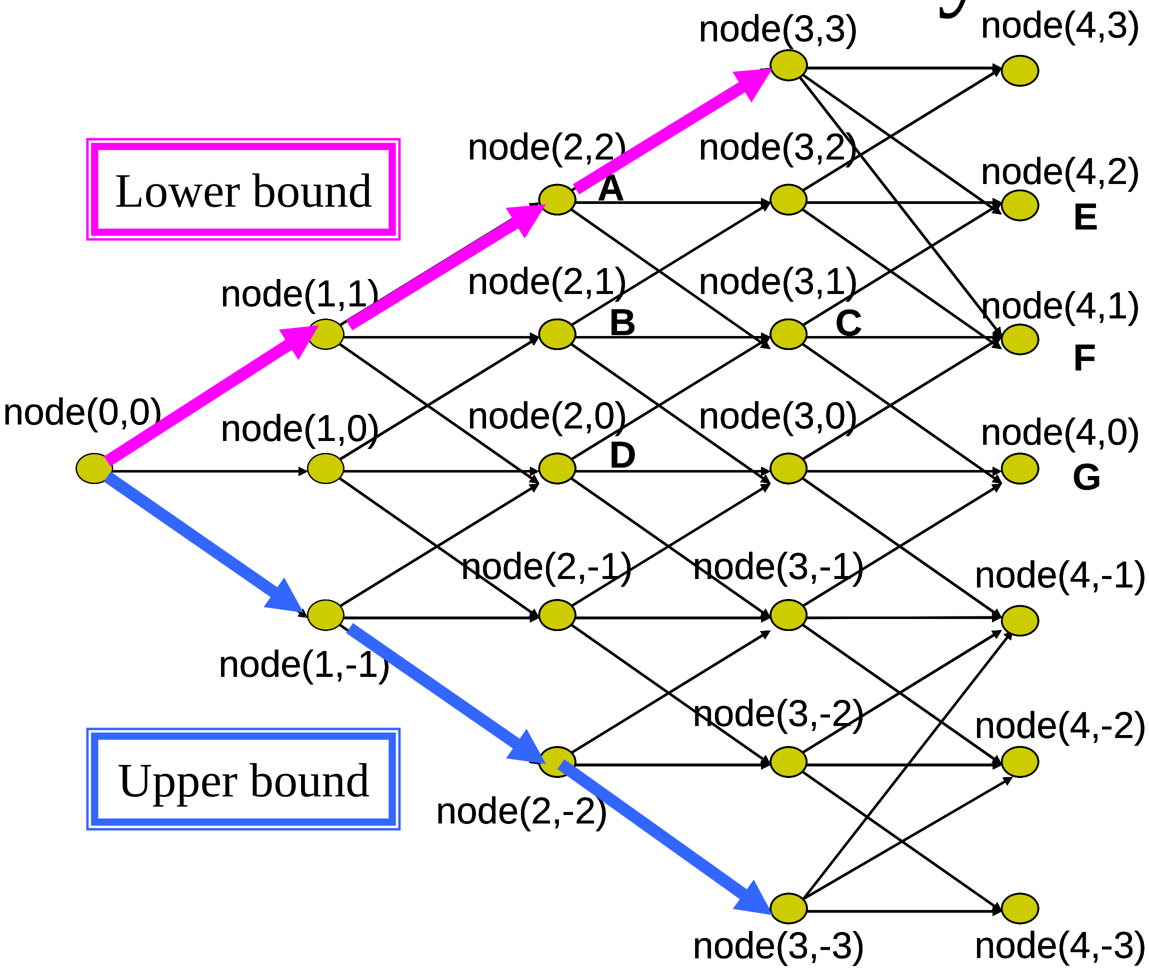
$$\text{Upper} \le 1 + 2 + \ldots + n - 1 = \frac{n(n-1)}{2}$$

$$\text{Lower} \ge -1 - 2 - \ldots - (n-1) = \frac{-n(n-1)}{2}$$

- 每個節點最多需n(n−1)+1 個狀態變數
- 樹的節點個數上限= $1 + 3 + \ldots + (2n+1) = \frac{(n+1)(2n+2)}{2}$

$$\frac{(n+1)(2n+2)}{2} \times (n \times (n-1) + 1) \Rightarrow O(n^4)$$ 電腦可處理

Hull-White Preliminary Tree



- The node (i, j) in the Hull-White preliminary tree means that at time $i\Delta t$, the rate is $j\Delta R$, where $\Delta R = \sigma\sqrt{3\Delta t}$

找出 f 的上下限

Example

node A : $Sum(2,2) = \{-3\}$
node B : $Sum(2,1) = \{-1,-2\}$
node D : $Sum(2,0) = \{1,0,-1\}$

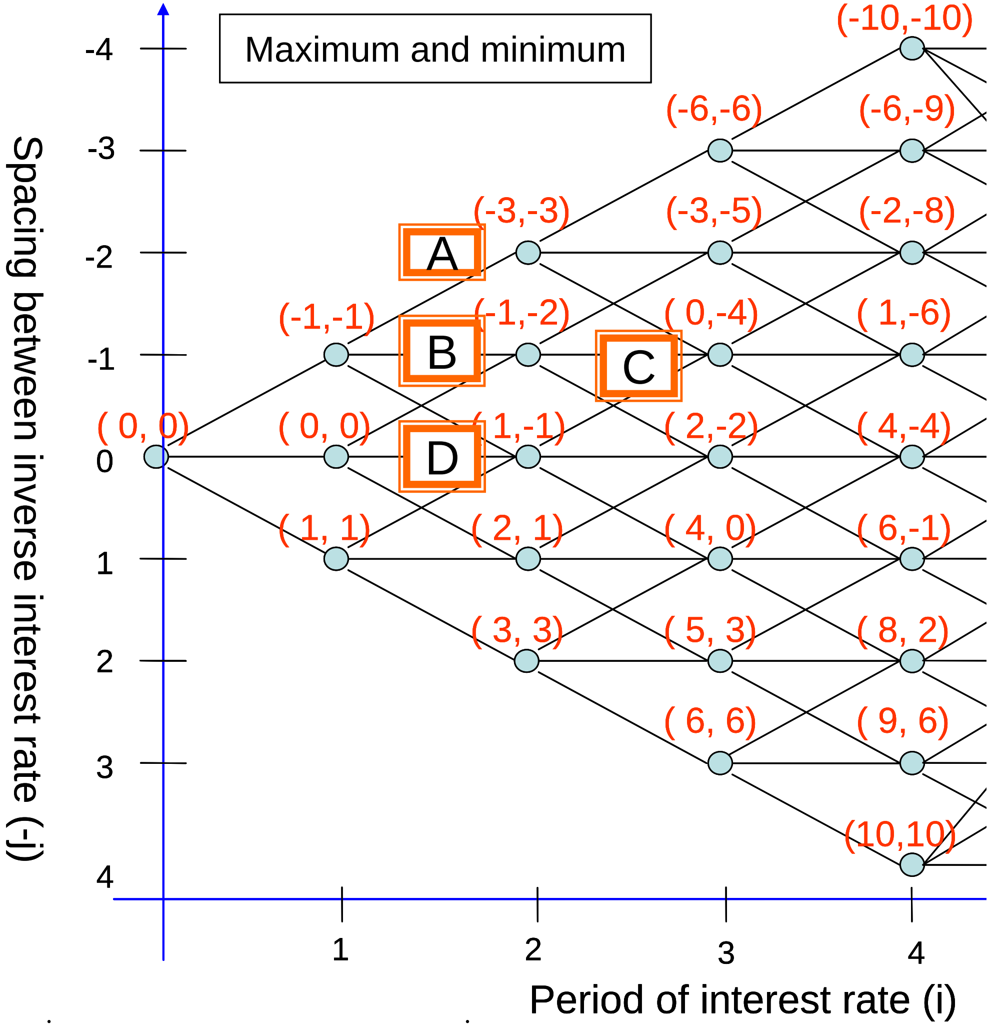
$$Min(Sum(2,j')) - 1 \le x \le Max(Sum(2,j')) - 1$$

$$where\ x \in Sum(3,1)\ j' \in \{0,1,2\}$$

$$\Rightarrow Min(Sum(2,2)) - 1 \le x \le Max(Sum(2,0)) - 1$$

$$\Rightarrow -4 \le x \le 0$$

$$\therefore Sum(3,1) = \{0,-1,-2,-3,-4\}$$



$$C_{3,1} = \{\sum_{k=1}^{i}(S_k - \alpha_k) - 0 * \Delta R,$$
$$\sum_{k=1}^{i}(S_k - \alpha_k) - 1\Delta R, \sum_{k=1}^{i}(S_k - \alpha_k) - 2\Delta R, \sum_{k=1}^{i}(S_k - \alpha_k) - 3\Delta R, \sum_{k=1}^{i}(S_k - \alpha_k) - 4\Delta R\}$$

Freeze at zero 限制

- 本期的債券利率=上期債券利率+Inverse floater
  - 當利率<0 →債券利率重設為0
- 第i 期利率可能無法寫成 $C_0+\sum_{j=1}^{i} S_j-\sum_{j=1}^{i}\alpha_j-f\Delta R$
  - Ex:

$$C_1+S_2-r_2<0 \rightarrow C_2=\left(C_1+S_2-r_2\right)^+=0$$

$$C_3=\left(S_3-r_3\right)^+$$

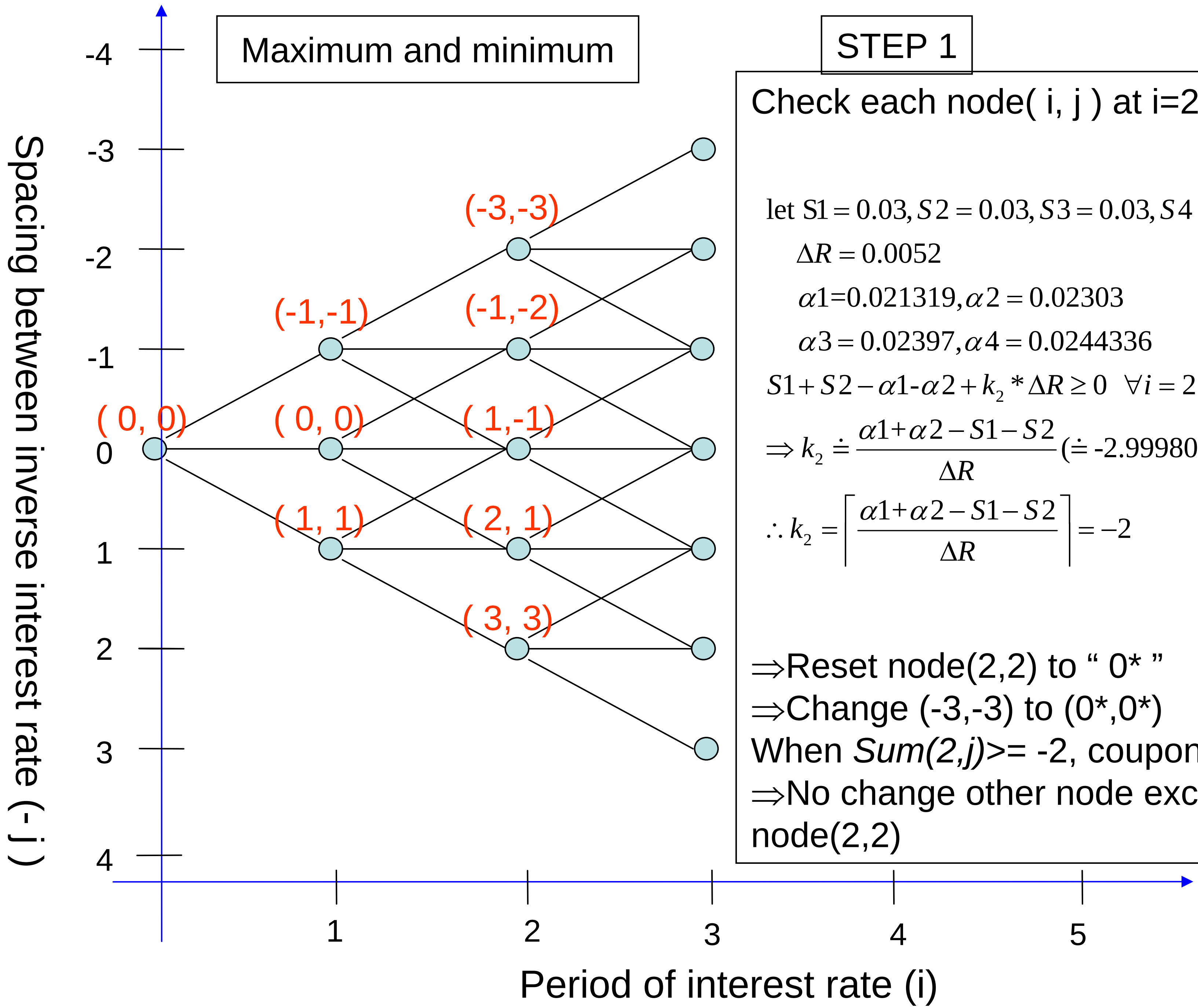
- 處理方法:
  - 拿掉 $C_0+\sum_{j=1}^{i} S_j-\sum_{j=1}^{i} R_j-f\Delta R$ 中小於0的狀態
  - 加入狀態0* →債券利率為0的Case
  - 使用內插計算0*所對應的價格

在Freeze at zero 限制下，找出f的上下限

- Suppose C0=0%
- Step 1:拿掉小於0的狀態, 加入狀態0*

$$\exists k_i \in \mathrm{Z} \;\; s.t. \;\; \sum_{k=1}^{i}(S_k - \alpha_k) + k_i \Delta R \geq 0$$

$$\Rightarrow k_i \geq \frac{\sum_{k=1}^{i}(S_k - \alpha_k)}{\Delta R}, \; k_i = \left\lceil \frac{\sum_{k=1}^{i}(S_k - \alpha_k)}{\Delta R} \right\rceil$$



STEP 1

Check each node(i, j) at i=2

$$\text{let } S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03$$

$$\Delta R = 0.0052$$

$$\alpha 1 = 0.021319, \alpha 2 = 0.02303$$

$$\alpha 3 = 0.02397, \alpha 4 = 0.0244336$$

$$S1 + S2 - \alpha 1 - \alpha 2 + k_2 * \Delta R \geq 0 \quad \forall i = 2$$

$$\Rightarrow k_2 \doteq \frac{\alpha 1 + \alpha 2 - S1 - S2}{\Delta R} (\doteq -2.999807692)$$

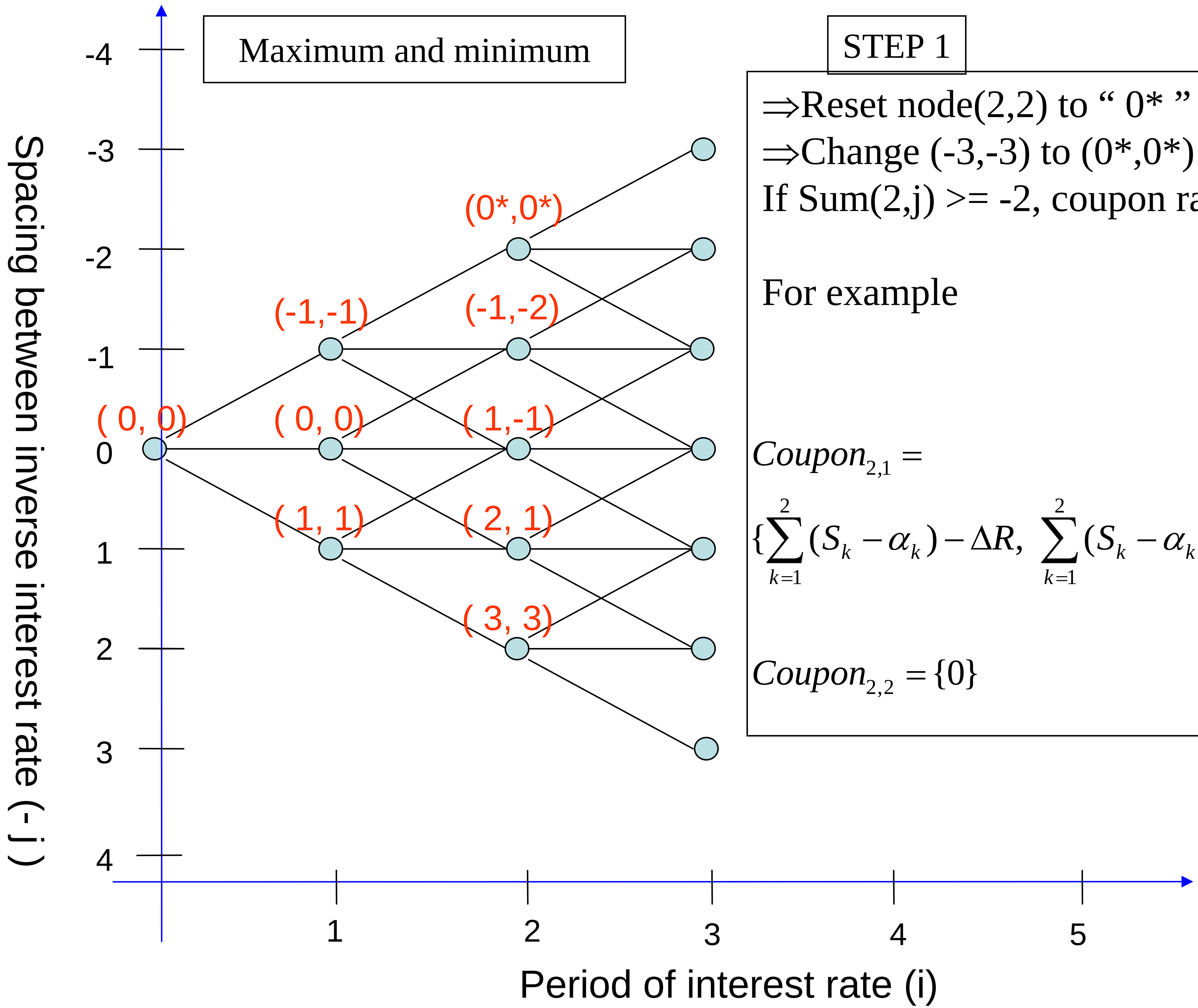
$$\therefore k_2 = \left\lceil \frac{\alpha 1 + \alpha 2 - S1 - S2}{\Delta R} \right\rceil = -2$$

⇒Reset node(2,2) to “ 0* ”

⇒Change (-3,-3) to (0*,0*)

When *Sum(2,j)*>= -2, coupon rate>0

⇒No change other node except node(2,2)



STEP 1

⇒Reset node(2,2) to " 0* "
⇒Change (-3,-3) to (0*,0*)
If Sum(2,j) >= -2, coupon rate>0

For example

$$Coupon_{2,1} =$$

$$\{\sum_{k=1}^{2}(S_k - \alpha_k) - \Delta R,\ \sum_{k=1}^{2}(S_k - \alpha_k) - 2\Delta R\}$$

$$Coupon_{2,2} = \{0\}$$

在Freeze at zero 限制下，找出f的上下限

- Suppose C0=0%
- Step 2:計算前一個node有0*的狀態

$$\exists \delta_{i,j} \in \mathrm{Z} \;\; s.t. \;\; \sum_{k=1}^{i} (S_k - \alpha_k) + \delta_{i,j} \Delta R \approx z_{i,j}$$

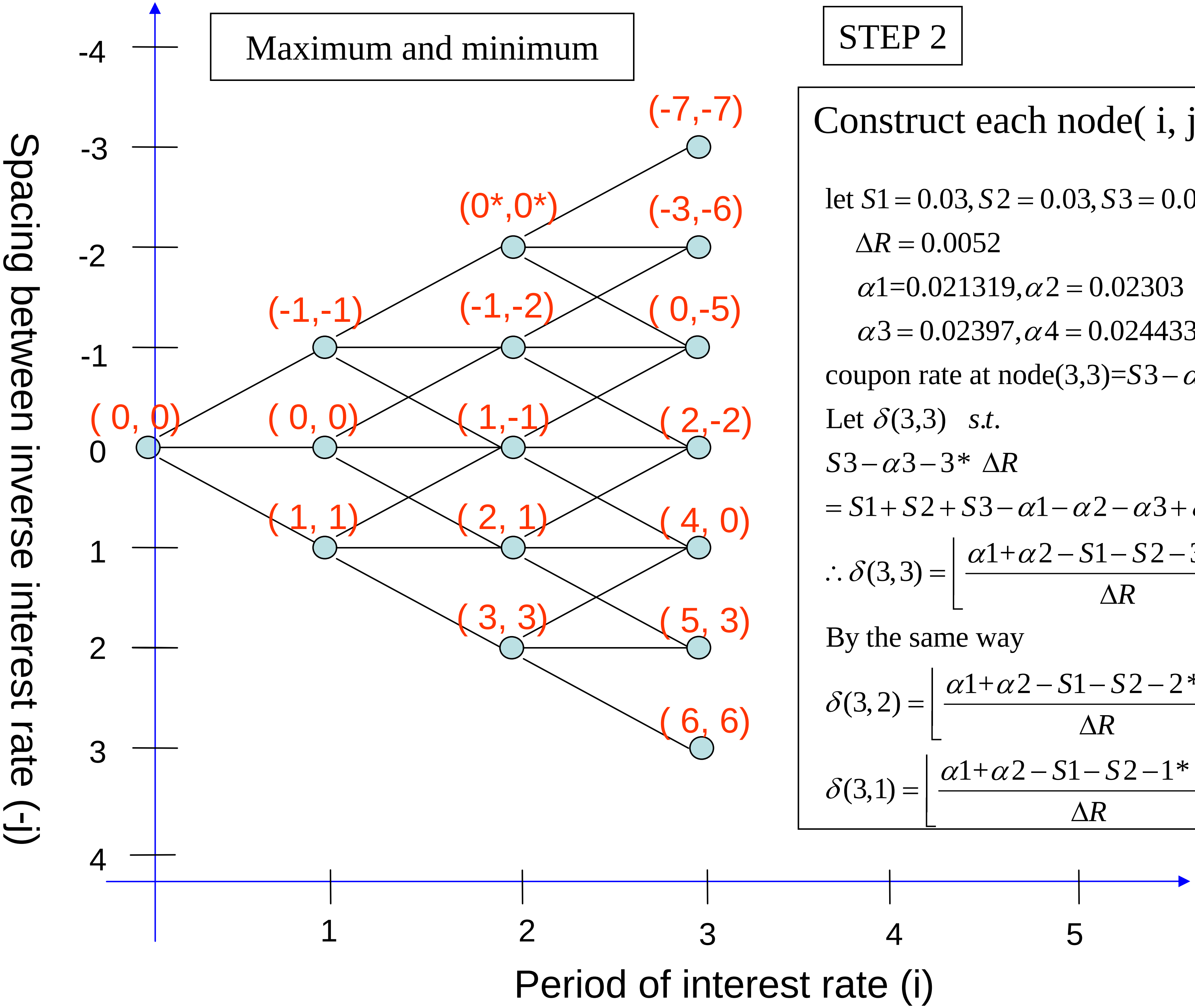
$$where \;\; \mathrm{z}_{i,j} = S_i - (\alpha_i + f_{i,j}) = (\mathrm{S}_i - \alpha_i) - j\Delta R$$

$$\Rightarrow \delta_{i,j} = \left\lfloor -\left(\frac{\sum_{k=1}^{i-1} (S_k - \alpha_k) + j\Delta R}{\Delta R}\right) \right\rfloor$$

- 使重設的利率也可寫成$\left(\mathrm{C}_0 + \sum_{k=1}^{i} S_k - \sum_{k=1}^{i} \alpha_k - f\,\Delta R \right)$的形式
  取下高斯=>為了利用內插法計算債券價值

Maximum and minimum

STEP 2



Construct each node(i, j) at i=3

let $S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03$

$\Delta R = 0.0052$

$\alpha 1 = 0.021319, \alpha 2 = 0.02303$

$\alpha 3 = 0.02397, \alpha 4 = 0.0244336$

coupon rate at node(3,3)=$S3 - \alpha 3 - 3 * \Delta R$

Let $\delta(3,3)$ $s.t.$

$S3 - \alpha 3 - 3 * \Delta R$

$= S1 + S2 + S3 - \alpha 1 - \alpha 2 - \alpha 3 + \delta(3,3) * \Delta R$

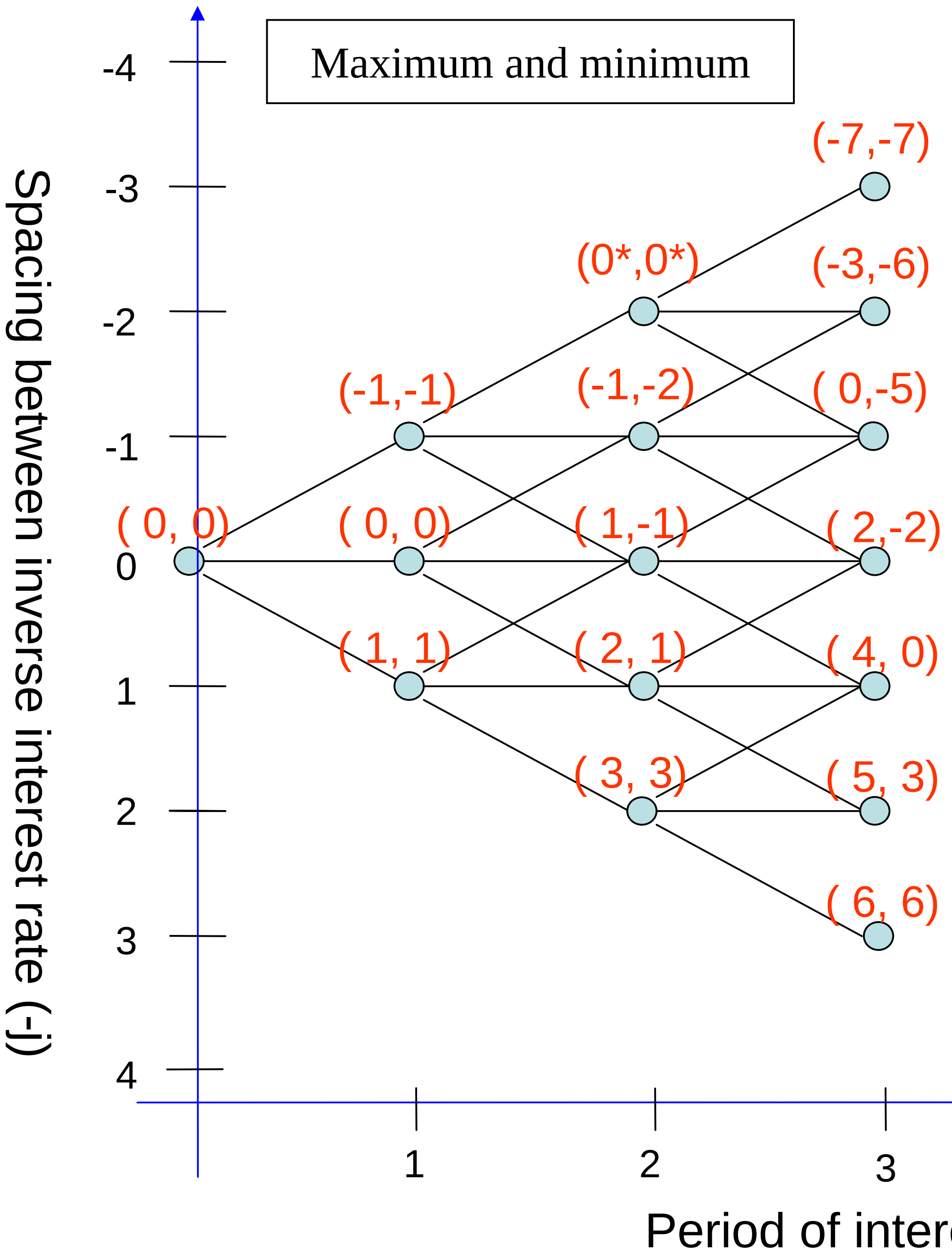
$$\therefore \delta(3,3) = \left\lfloor \frac{\alpha 1 + \alpha 2 - S1 - S2 - 3 * \Delta R}{\Delta R} \right\rfloor = -7$$

By the same way

$$\delta(3,2) = \left\lfloor \frac{\alpha 1 + \alpha 2 - S1 - S2 - 2 * \Delta R}{\Delta R} \right\rfloor = -6$$

$$\delta(3,1) = \left\lfloor \frac{\alpha 1 + \alpha 2 - S1 - S2 - 1 * \Delta R}{\Delta R} \right\rfloor = -5$$

Maximum and minimum



STEP 3

Construct each node(i, j) at i=3

let $S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03$

$\Delta R = 0.0052$

$\alpha 1 = 0.021319, \alpha 2 = 0.02303$

$\alpha 3 = 0.02397, \alpha 4 = 0.0244336$

$S1 + S2 + S3 - \alpha 1 - \alpha 2 - \alpha 3 + k_3 * \Delta R > 0 \quad \forall N = 3$

$$\Rightarrow k_3 > \frac{\alpha 1 + \alpha 2 + \alpha 3 - S1 - S2 - S3}{\Delta R}$$

$$\therefore k_3 = \left\lceil \frac{\alpha 1 + \alpha 2 + \alpha 3 - S1 - S2 - S3}{\Delta R} \right\rceil = -4$$

Reset node(3,3), node(3,2), and node(3,1).

Change (-7,-7) to (0*,0*)

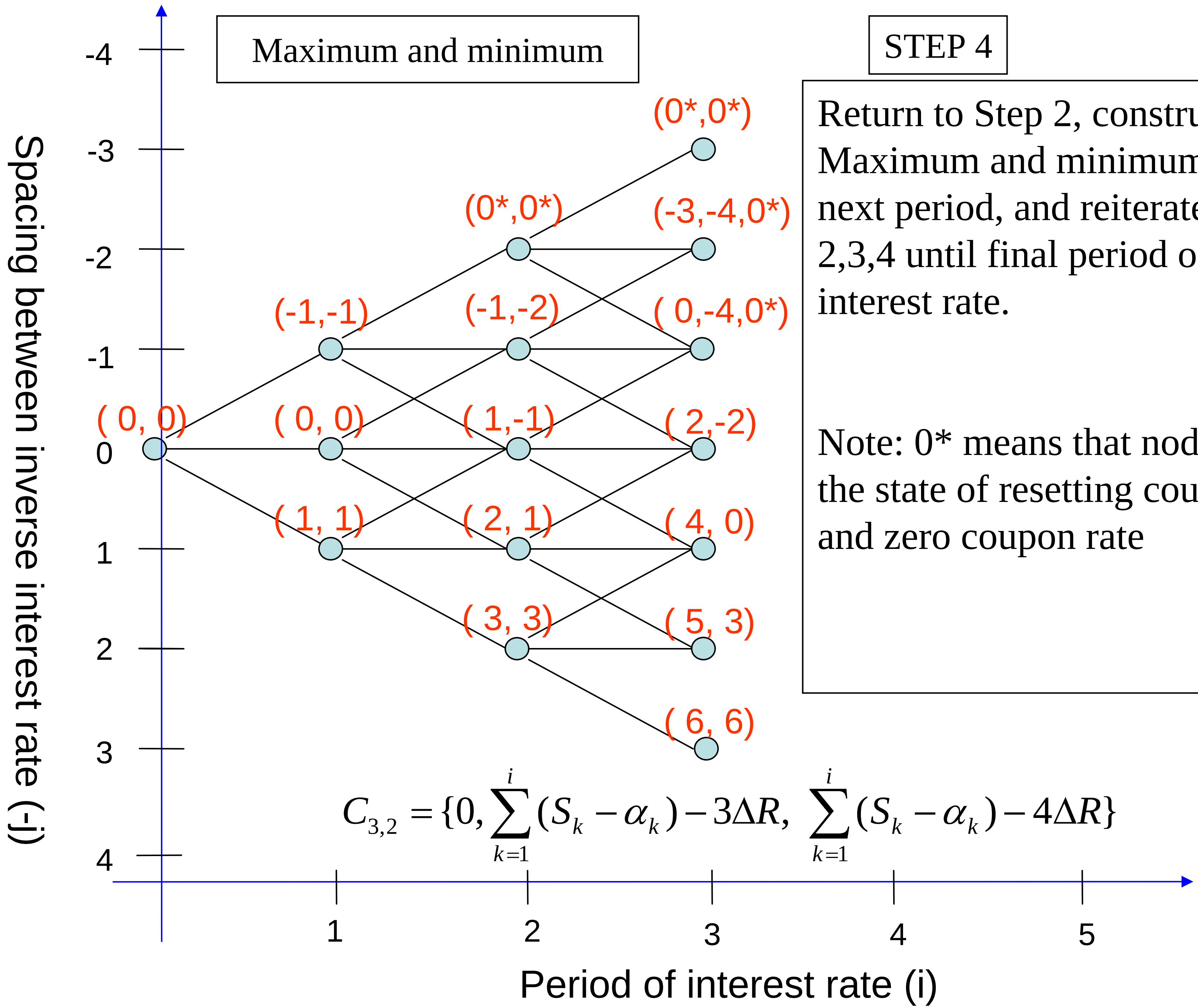
(-3,-6) to (-3,-4,0*)

(0,-5) to (0,-4,0*)

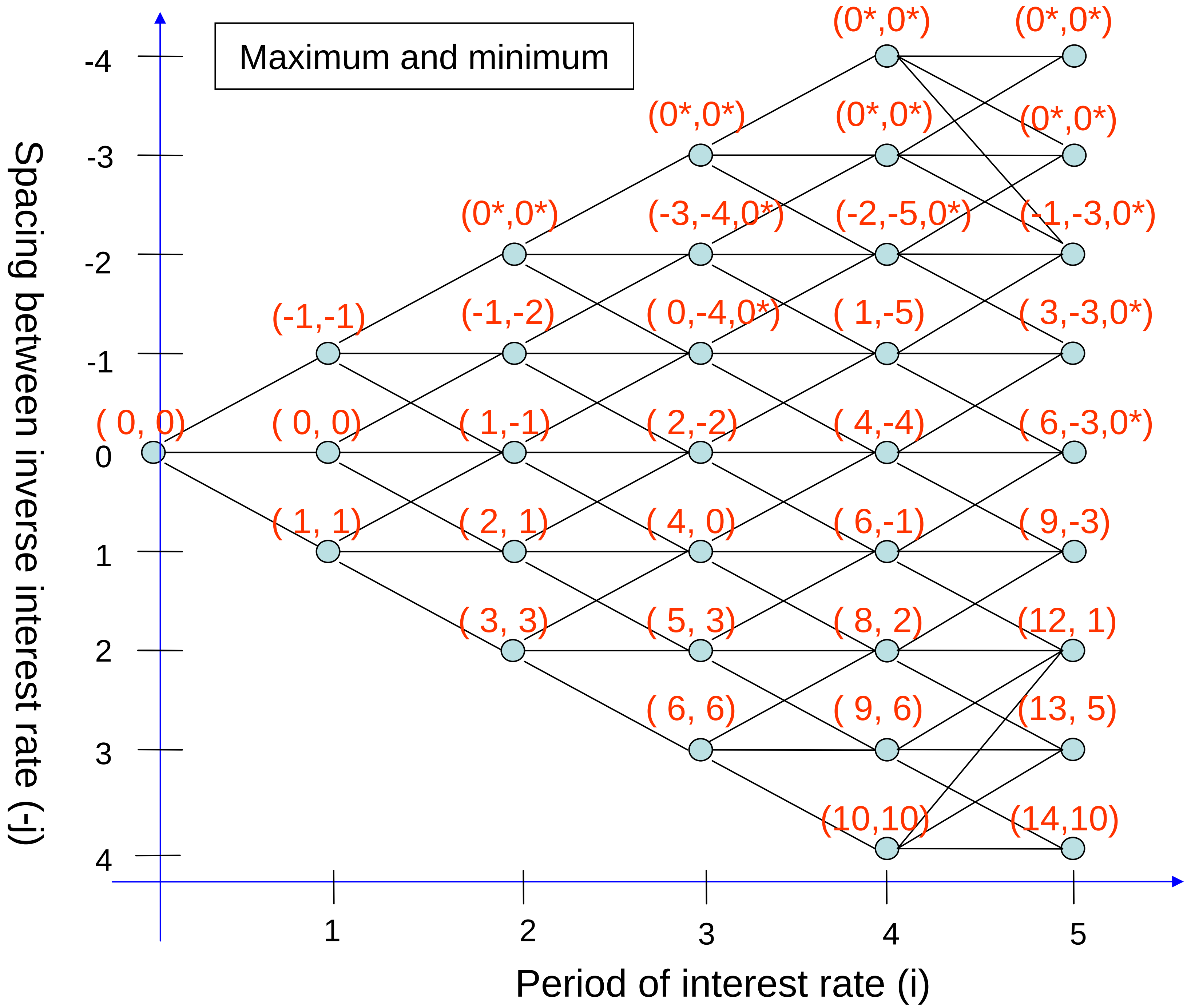
STEP 4

Return to Step 2, construct the Maximum and minimum of next period, and reiterate Step 2,3,4 until final period of interest rate.

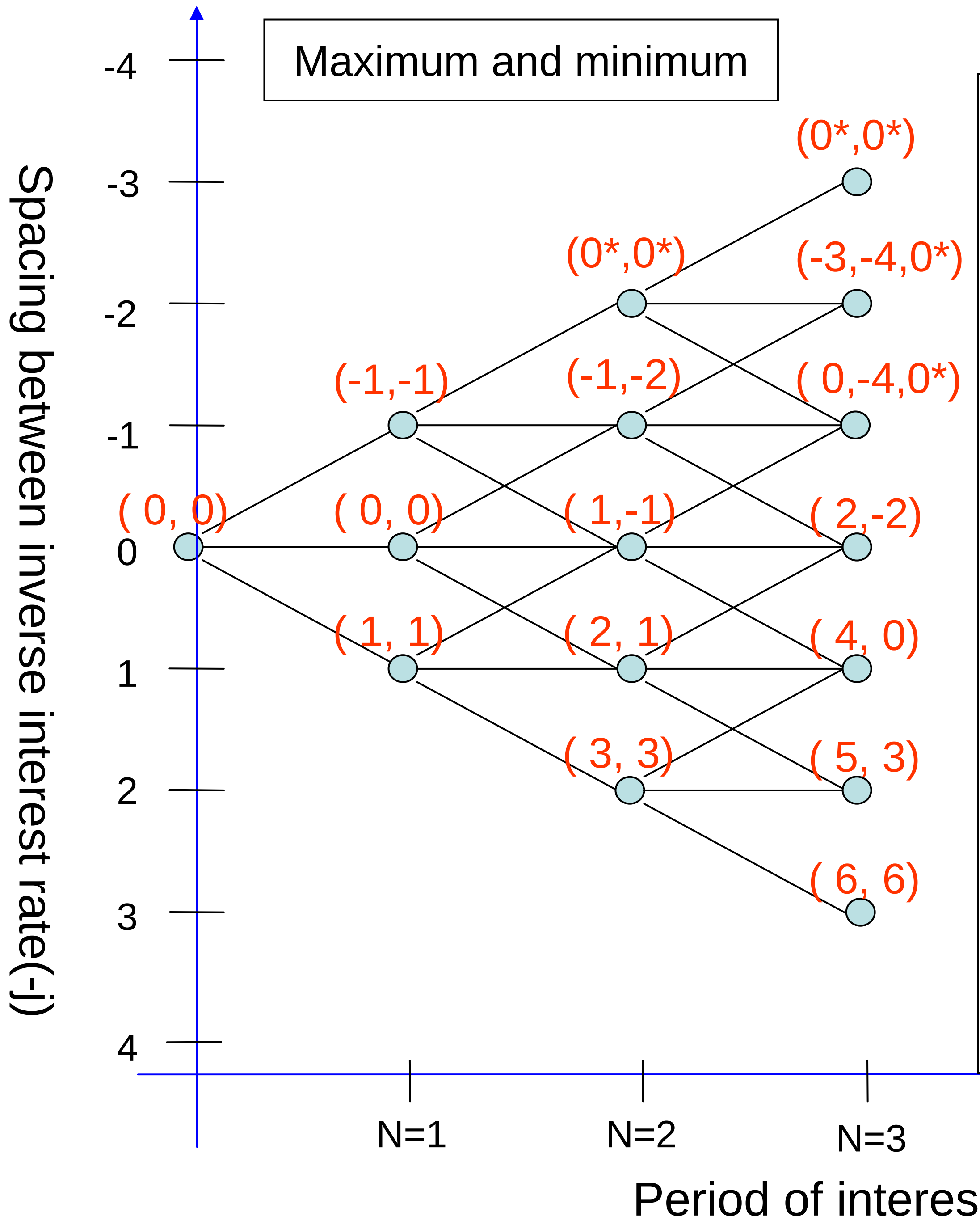
Note: 0* means that node have the state of resetting coupon rate and zero coupon rate



$$C_{3,2} = \{0, \sum_{k=1}^{i} (S_k - \alpha_k) - 3\Delta R,\ \sum_{k=1}^{i} (S_k - \alpha_k) - 4\Delta R\}$$


Maximum and minimum
Spacing between inverse interest rate (-j)
Period of interest rate (i)
-4
-3
-2
-1
0
1
2
3
4
1
2
3
4
5
(0, 0)
(-1,-1)
(0, 0)
(1, 1)
(0*,0*)
(-1,-2)
(1,-1)
(2, 1)
(3, 3)
(0*,0*)
(-3,-4,0*)
(0,-4,0*)
(2,-2)
(4, 0)
(5, 3)
(6, 6)
(0*,0*)
(0*,0*)
(-2,-5,0*)
(1,-5)
(4,-4)
(6,-1)
(8, 2)
(9, 6)
(10,10)
(0*,0*)
(0*,0*)
(-1,-3,0*)
(3,-3,0*)
(6,-3,0*)
(9,-3)
(12, 1)
(13, 5)
(14,10)

Maximum and minimum



STEP 5

Save possible state of each node in the same period.
ex. (N=3)

Node (N, j)	Node (3,3)
Maximum	X
Minimum	X
0*(1 or 0)	1
total	1

Node (3, 2)	Node (3, 0)
-3	2
-4	-2
1	0
3	5

0*
-3
-4
0*
0
⋮
-4
0*
2
⋮
-2
4
⋮
0
⋮

計算snowball債券價值

B (i, j, Sum(a)) means bond value at *node (i, j)* with *Sum (i, j)=a.*

Example 1:

$$Sum(a_u) = Sum(-4) - j_u = -4 - 2 = -6 < k_4$$

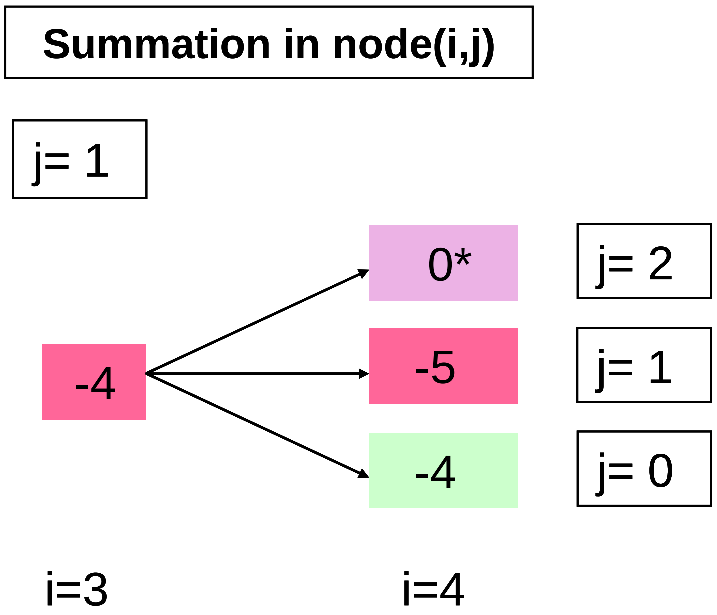
$$Sum(a_m) = Sum(-4) - j_m = -4 - 1 = -5$$

$$Sum(a_d) = Sum(-4) - j_d = -4 - 0 = -4$$

$$Sum(a_u) = Sum(0*)$$

$$Sum(a_m) = Sum(-5)$$

$$Sum(a_d) = Sum(-4)$$



$$D(3,1,Sum(-4))$$

$$= (P_u B(4,2,Sum(0*)) + P_m B(4,1,Sum(-5)) + P_d B(4,0,Sum(-4))) * \frac{1}{1 + (\alpha_3 + 1 * \Delta R) * (t_4 - t_3)}$$

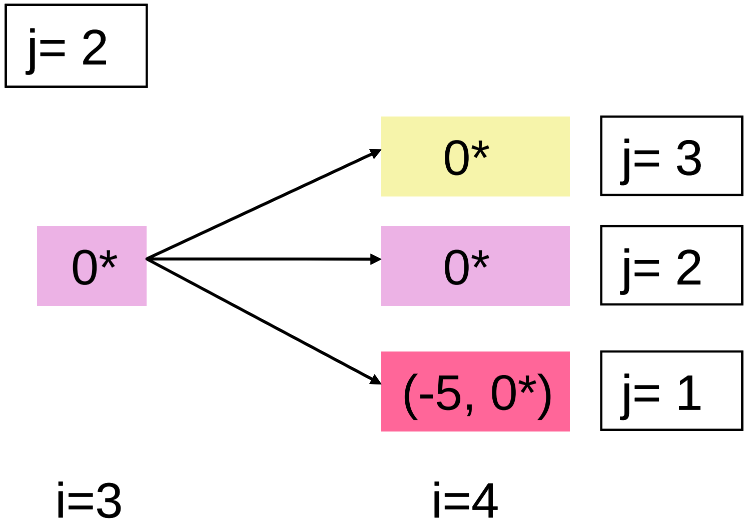
$$B(3,1,Sum(-4)) = \{\min(D(3,1,Sum(-4)),\ 1) + C;\ \forall C \in \{C_{2,j^*} : Sum(2, j^*) - 1 = -4,\ j^* = 0,1,2\}\ \}$$

用線性插補法求snowball價值(1)

Example 2

- Actual coupon rate $\chi_{i,j}$ fixed at *node (4, j'), j'={1,2,3}* is following:

Summation in node(i,j)



$$\chi_{i,j} = \max(S_i - (\alpha_i + f_{i,j}\Delta R)\ ,0)$$

let $S1 = 0.03, S2 = 0.03, S3 = 0.03, S4 = 0.03$

$\Delta R = 0.0052$

$\alpha 1 = 0.021319, \alpha 2 = 0.02325$

$\alpha 3 = 0.02397, \alpha 4 = 0.0244336$

$\chi_{4,3} = \max(S_4 - \alpha_4 - 3\Delta R\ ,0) = 0$

$\chi_{4,2} = \max(S_4 - \alpha_4 - 2\Delta R\ ,0) = 0$

$\chi_{4,1} = \max(S_4 - \alpha_4 - 1\Delta R\ ,0) = 0.0003664$

$Sum(a_u) = Sum(0^*)$

$Sum(a_m) = Sum(0^*)$

$Sum(a_d) \in (Sum(-5), Sum(0^*))$

Let k'' s.t. $S4 - \alpha 4 - 1 * \Delta R = \sum_{k=1}^{4}(S_k - \alpha_k) + k'' * \Delta R$

$k'' = -5.169423077$

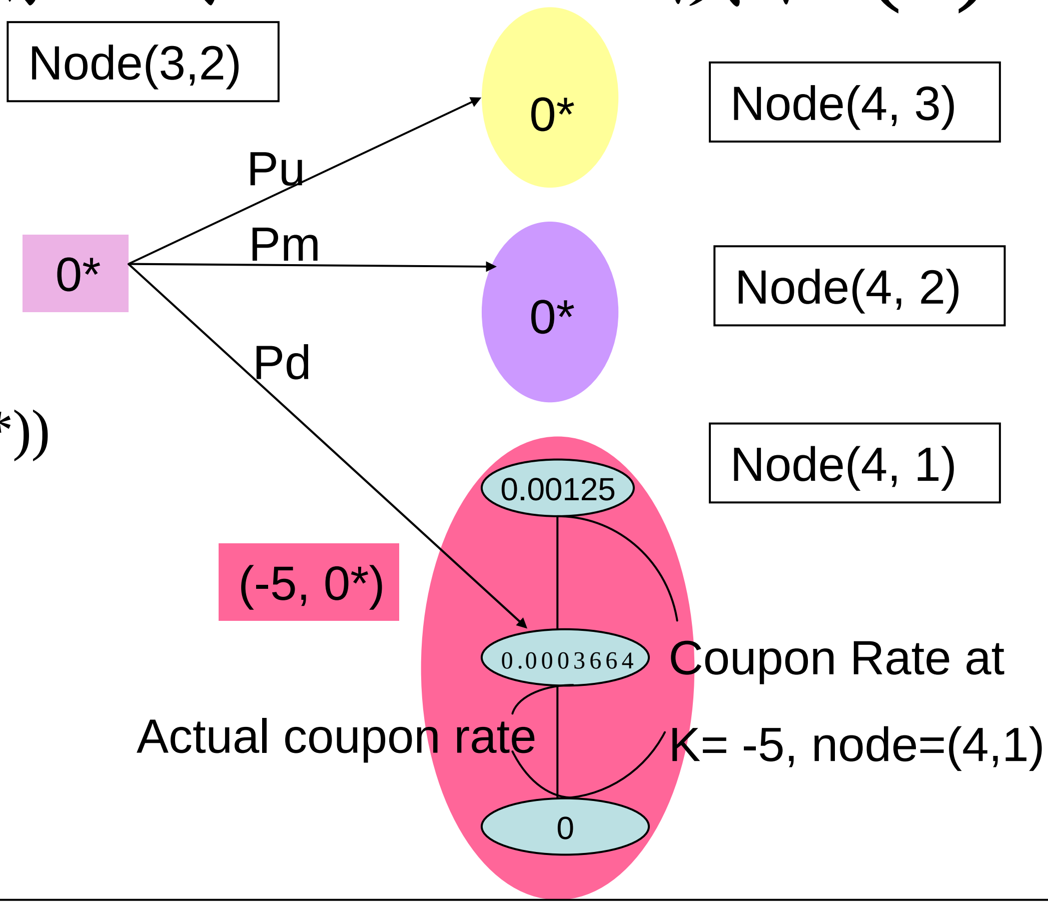
$\therefore$ Actual coupon rate $\chi_{4,1} \in (C_{4,1,Sum(-5)}, C_{4,1,Sum(0^*)})$

用線性插補法求snowball價值(2)

$$Sum(a_u) = Sum(0^*)$$

$$Sum(a_m) = Sum(0^*)$$

$$Sum(a_d) \in (Sum(-5), Sum(0^*))$$



Linear Interpolation

Actual discounted bond value from node(4,1) is calculated by interpolation method :

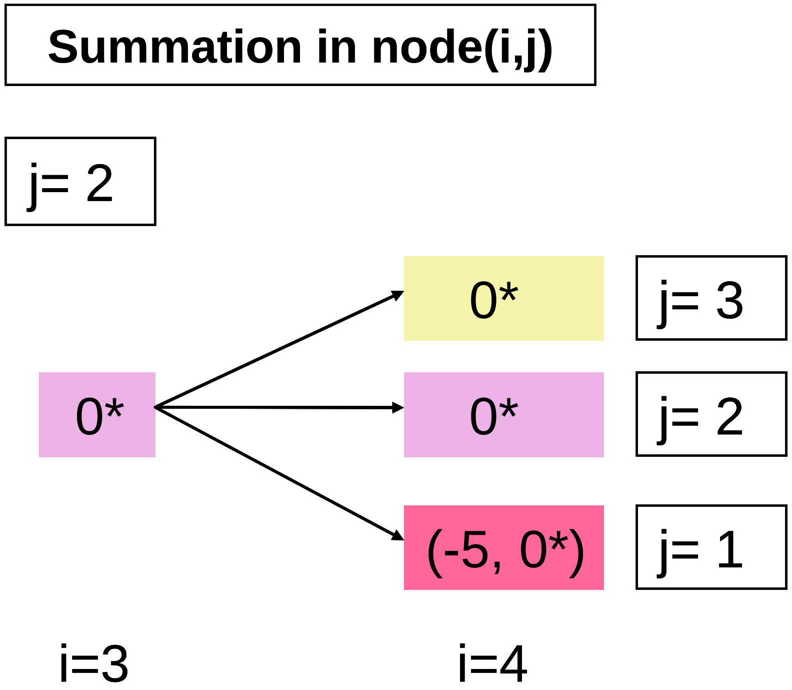
$C_{4,1,Sum(-5)}$: Actual coupon at node(4,1)

$= [B(4,1,Sum(-5)) - B(4,1,Sum(0^*))] : [\text{Actual bond value at node}(4,1) - B(4,1,Sum(0^*))]$

$\therefore$ Actual bond value at node(4,1)

$$= \frac{\text{Actual reset coupon at node}(4,1) * [B(4,1,Sum(-5)) - B(4,1,Sum(0^*))]}{C_{4,1,Sum(-5)}} + B(4,1,Sum(0^*))$$

用線性內插法求snowball價值(3)



$$D(3,2,Sum(0*)) = (P_u B(4,3,Sum(0*)) + P_m B(4,2,Sum(0*))$$

$$+ P_d * \text{Actual bond value}(4,1)) * \frac{1}{1+(\alpha_3 + 2*\Delta R)*(t_4 - t_3)}$$

$$B(3,2,Sum(0*)) = \{\min(D(3,2,Sum(0*)),\ 1) + C;\ \forall \mathrm{C} \in \{C_{2,j^*}: Sum(2,j^*) - 2 = 0*,\ j^* = 1,2\}\}$$

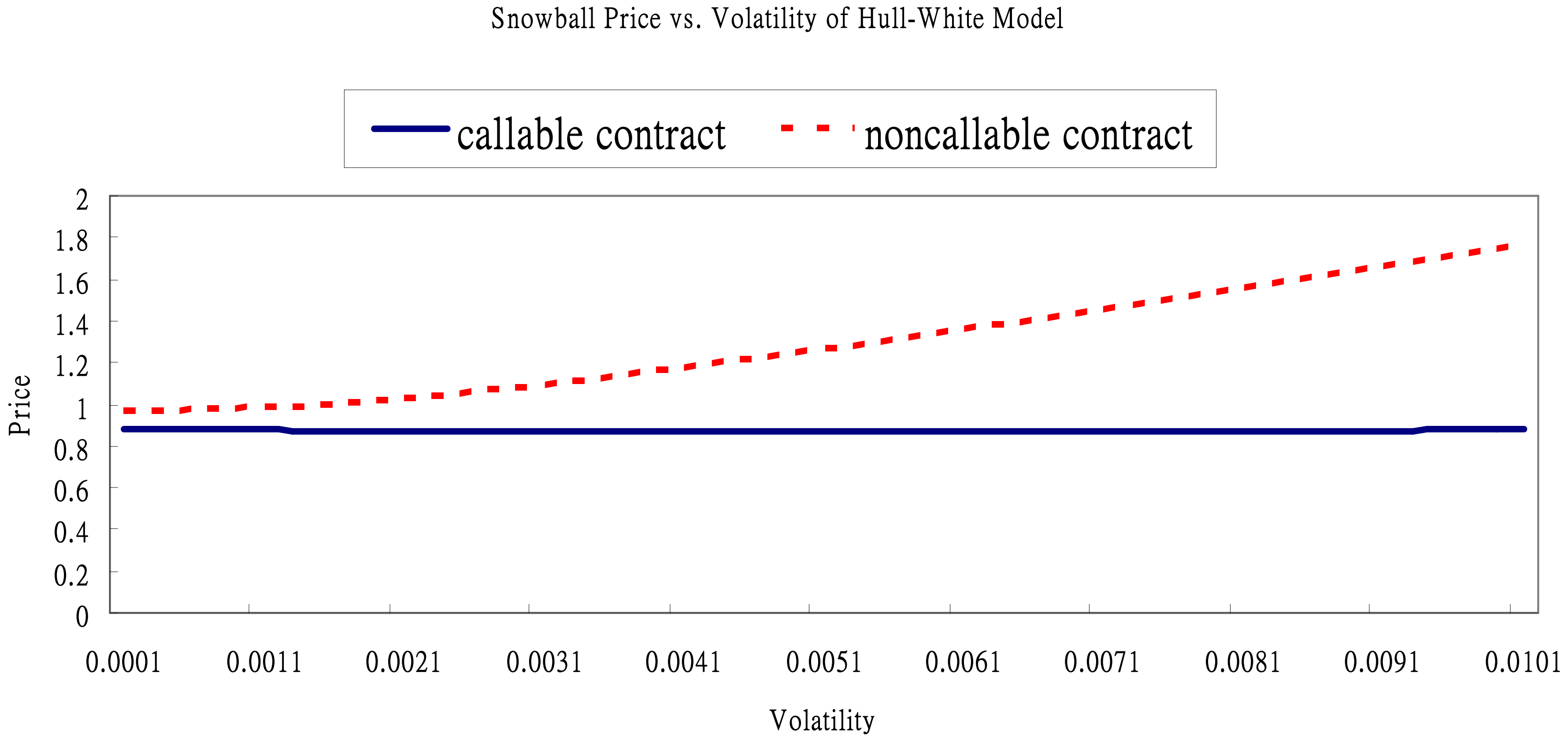
敏感度分析

- A snowball note issued by Bank SinoPac which the contract could be redeemed with par value after the third year.
- Cn, i : coupon rate at i quarter of n year.
- FR : the fixing rate of 90 days CP; if i-1=0, Cn, i-1= Cn-1, 4 for n=1…10, i=1..4.

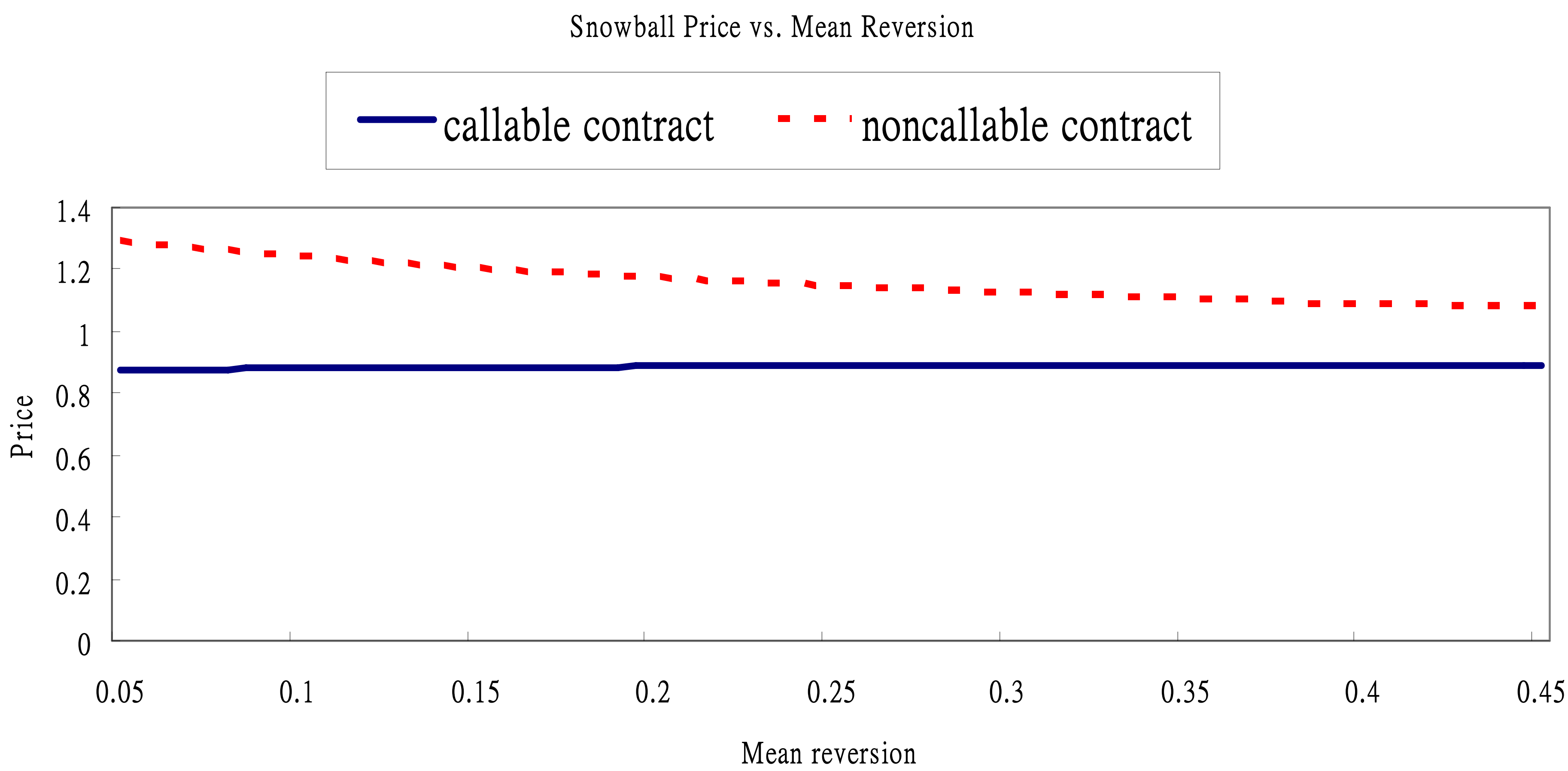
Year	Coupon rate (Cn,i)
1	C1,i=3%, i=1,2,3,4
2	C2,i=C2,i-1+1.40%-FR2,i
3	C3,i=C3,i-1+1.65%-FR3,i
4	C4,i=C4,i-1+1.90%-FR4,i
5	C5,i=C5,i-1+2.15%-FR5,i
6	C6,i=C6,i-1+2.40%-FR6,i
7	C7,i=C7,i-1+2.65%-FR7,i
8	C8,i=C8,i-1+2.90%-FR8,i
9	C9,i=C9,i-1+3.15%-FR9,i
10	C10,i=C10,i-1+3.40%-FR10,i

敏感度分析(1)
Price vs. Volatility

$$Coupon(i) = (Coupon(i-1) + Spread(i) - Floating\ rate(i))^{+}$$

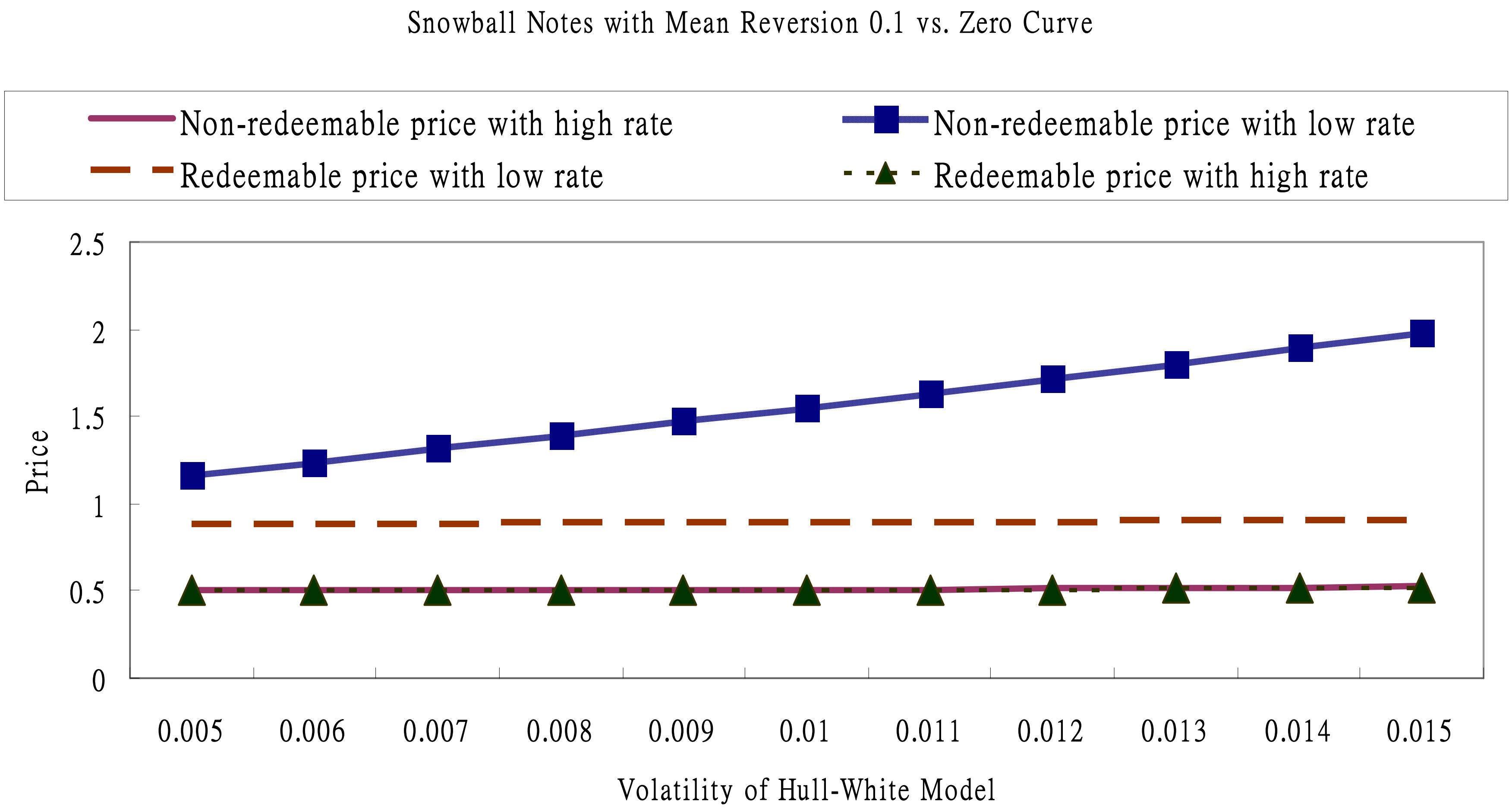


敏感度分析(2)
Price vs. Mean Reversion



敏感度分析(3)
Price vs. Market Rates

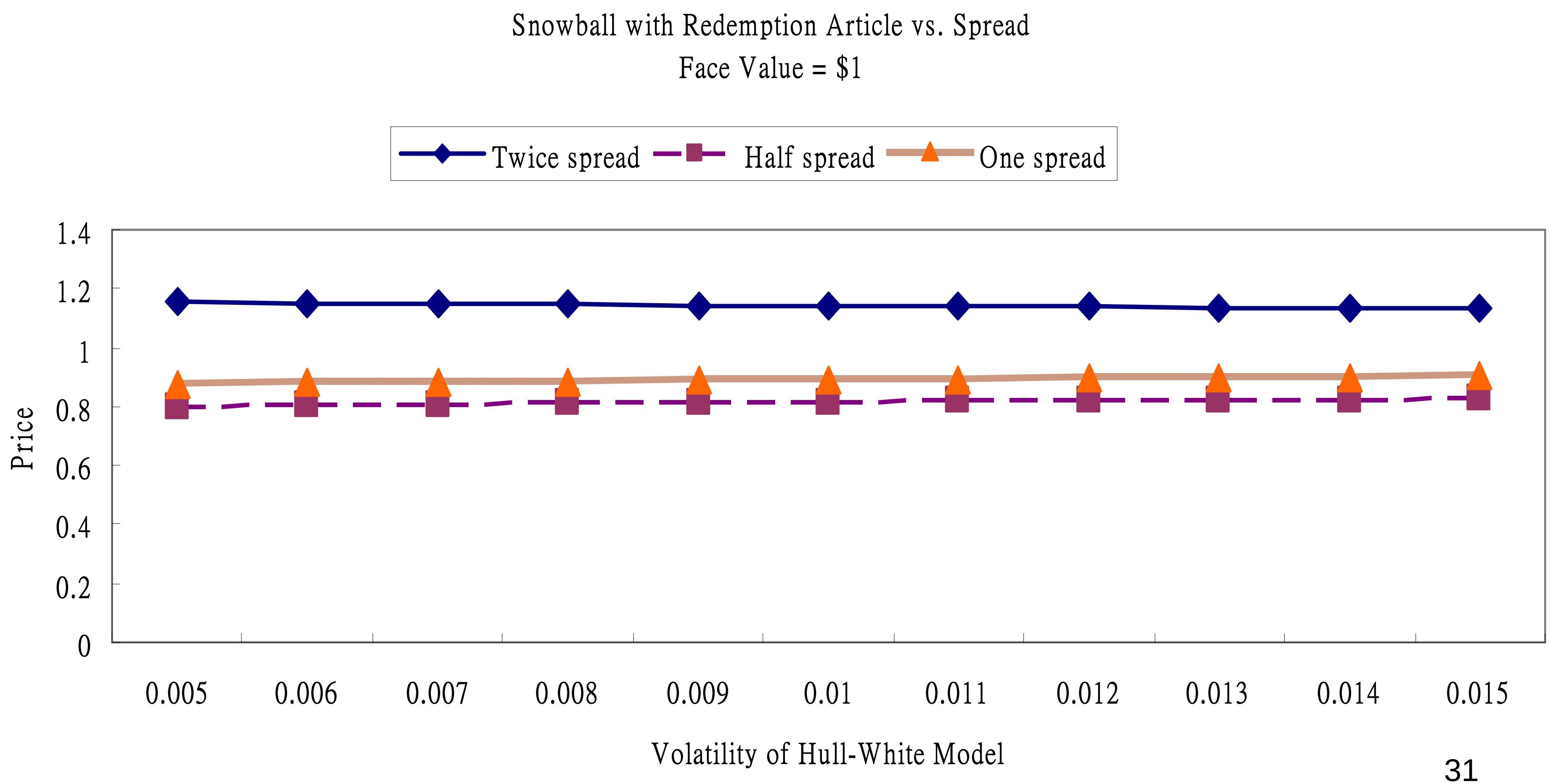
$$Coupon(i) = (Coupon(i-1) + Spread(i) - Floating\ rate(i))^{+}$$



敏感度分析(4)

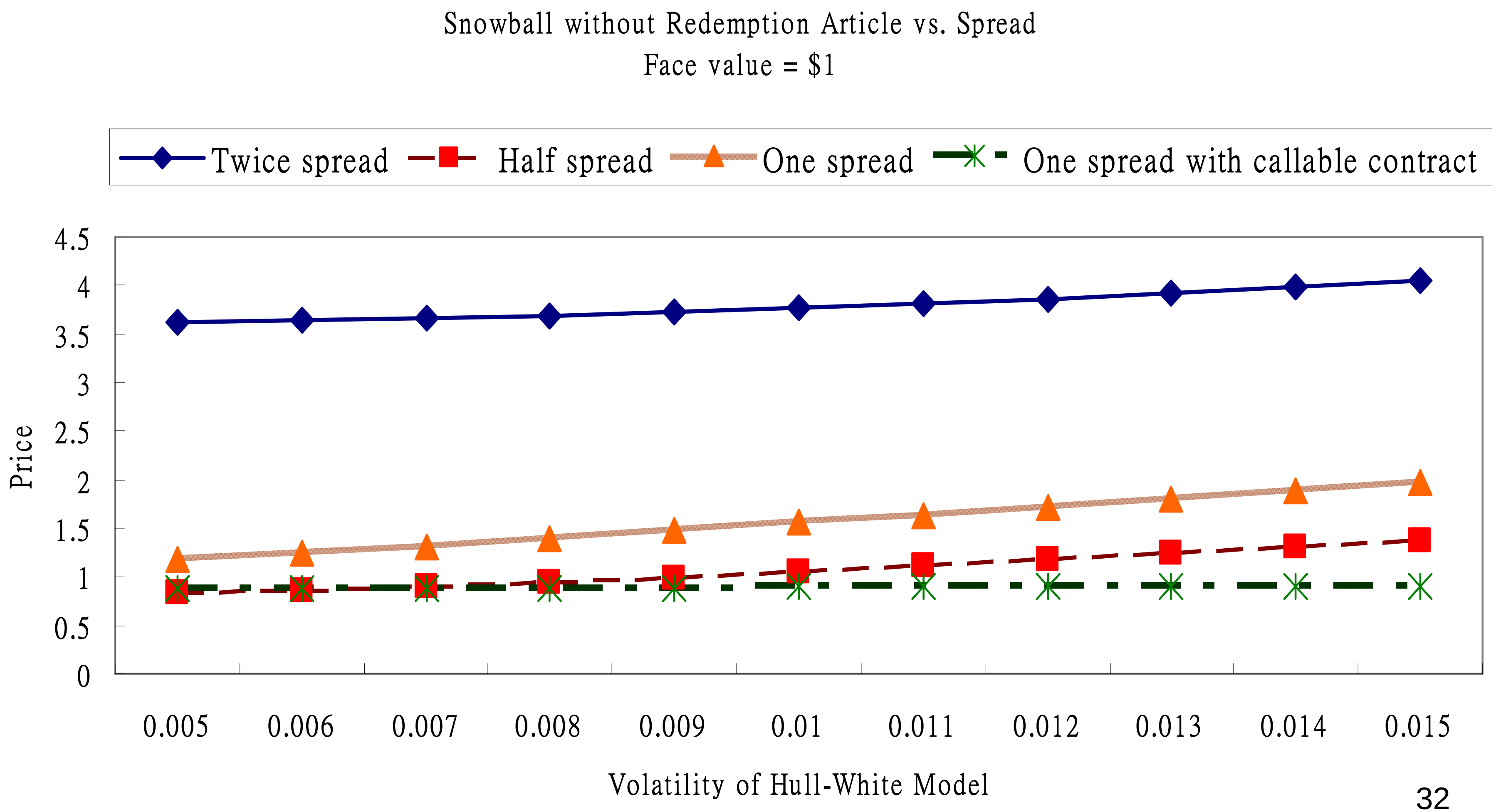
Redeemable Snowball Price vs. Spread Article

$$Coupon(i) = (Coupon(i-1) + Spread(i) - Floating\ rate(i))^{+}$$

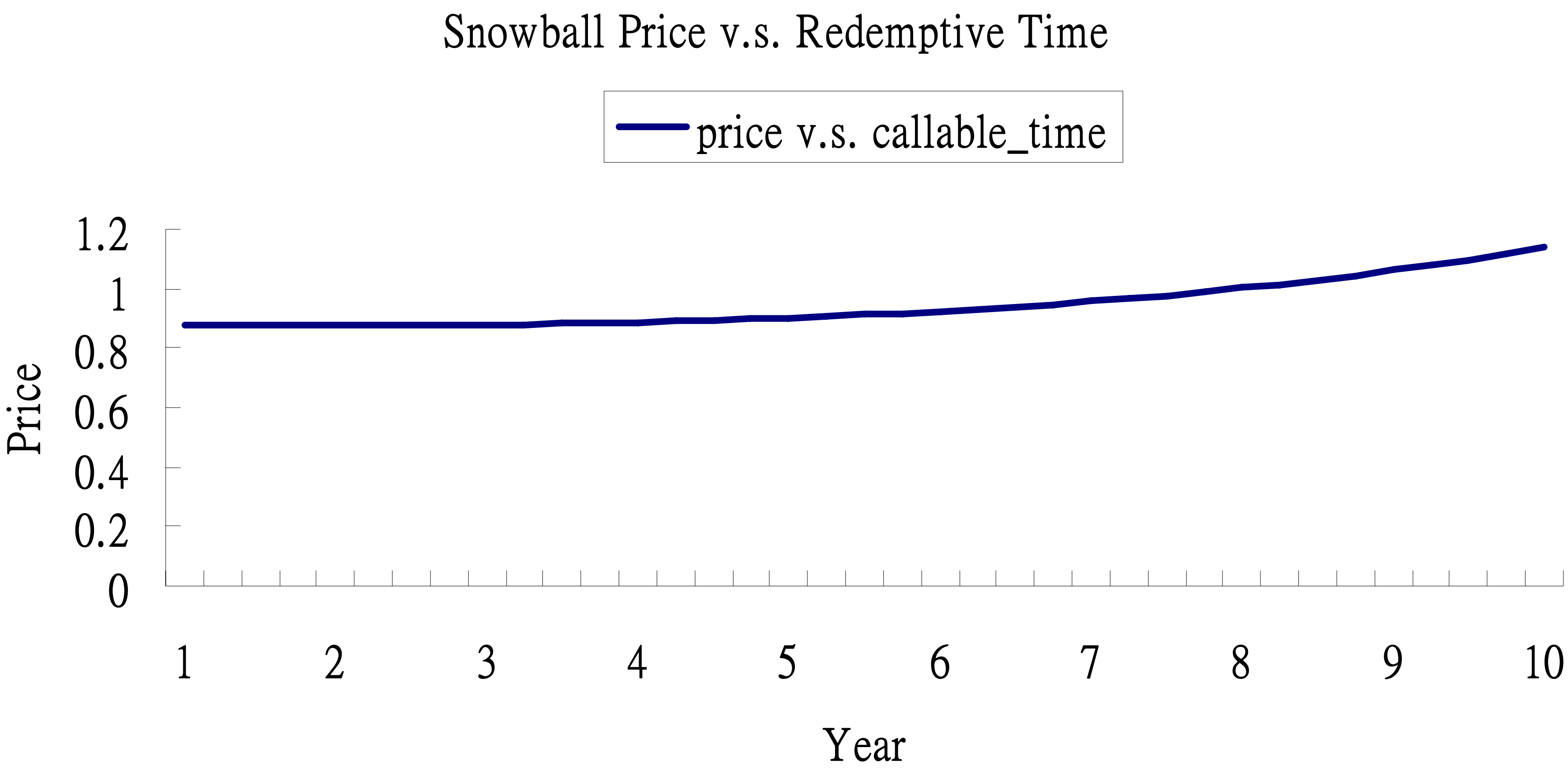


Non-redeemable Snowball Price vs. Spread Article

$$Coupon(i) = (Coupon(i-1) + Spread(i) - Floating\ rate(i))^{+}$$



敏感度分析(5)
Snowball Price vs. Redemptive Time



實證結果

- Zero rates data from CHINA BILLS FINANCE CORPORATION (中華票券)

quarter / year	1	2	3	4
1	1.4610%	1.5160%	1.5900%	1.6505%
2	1.7115%	1.7497%	1.7880%	1.8264%
3	1.8649%	1.8905%	1.9162%	1.9420%
4	1.9678%	1.9898%	2.0118%	2.0339%
5	2.0561%	2.0730%	2.0900%	2.1070%
6	2.1241%	2.1466%	2.1691%	2.1918%
7	2.2145%	2.2372%	2.2601%	2.2830%
8	2.3059%	2.3230%	2.3401%	2.3572%
9	2.3744%	2.3917%	2.4090%	2.4264%
10	2.4439%	2.4614%	2.4790%	2.4966%

估計Hull-White model參數

- Mean reversion and volatility are constants in Hull-White model.
- Use cap price to calibrate parameters.
- A popular goodness-of-fit measure is

$$SSE = \min_{a,\sigma} \sum_{k} \sum_{i=1}^{n} (U_{ik} - V_{ik})^2$$

$$k \in \{1.5\%, 2.5\%, 3.5\%, 4.5\%\}$$

where *Ui* is the market price calculated by Black's equation and *Vi* is the price given by the Hull-White model.

Cap is a portfolio of interest rate options of caplet

T_r : Total life of cap

K_{R_cap} : Cap rate

Δk : Period time, $\Delta k = t_{k+1} - t_k$

L : Principal

r_k : Interest rate for period between time t_k and t_{k+1}

F_t : Forward rate for period between time t_k and t_{k+1}

$$Caplet = L\Delta k \max(r_k - K_{R\_cap}, 0)$$

- Caplet present value by Black's equation

$$L\Delta k P(0, t_{k+1})[F_k N(d_1) - K_{R\_cap} N(d_2)]$$

$$d_1 = \frac{\ln[F_k / K_{R\_cap}] + \sigma_k{}^2 t_k / 2}{\sigma_k \sqrt{t_k}} \qquad d_2 = d_1 - \sigma_k \sqrt{t_k}$$

Cap is a portfolio of bond options

- Interest rate cap could be characterized as a portfolio of put options on zero-coupon bond.
- Caplet as a put options on zero-coupon bond:

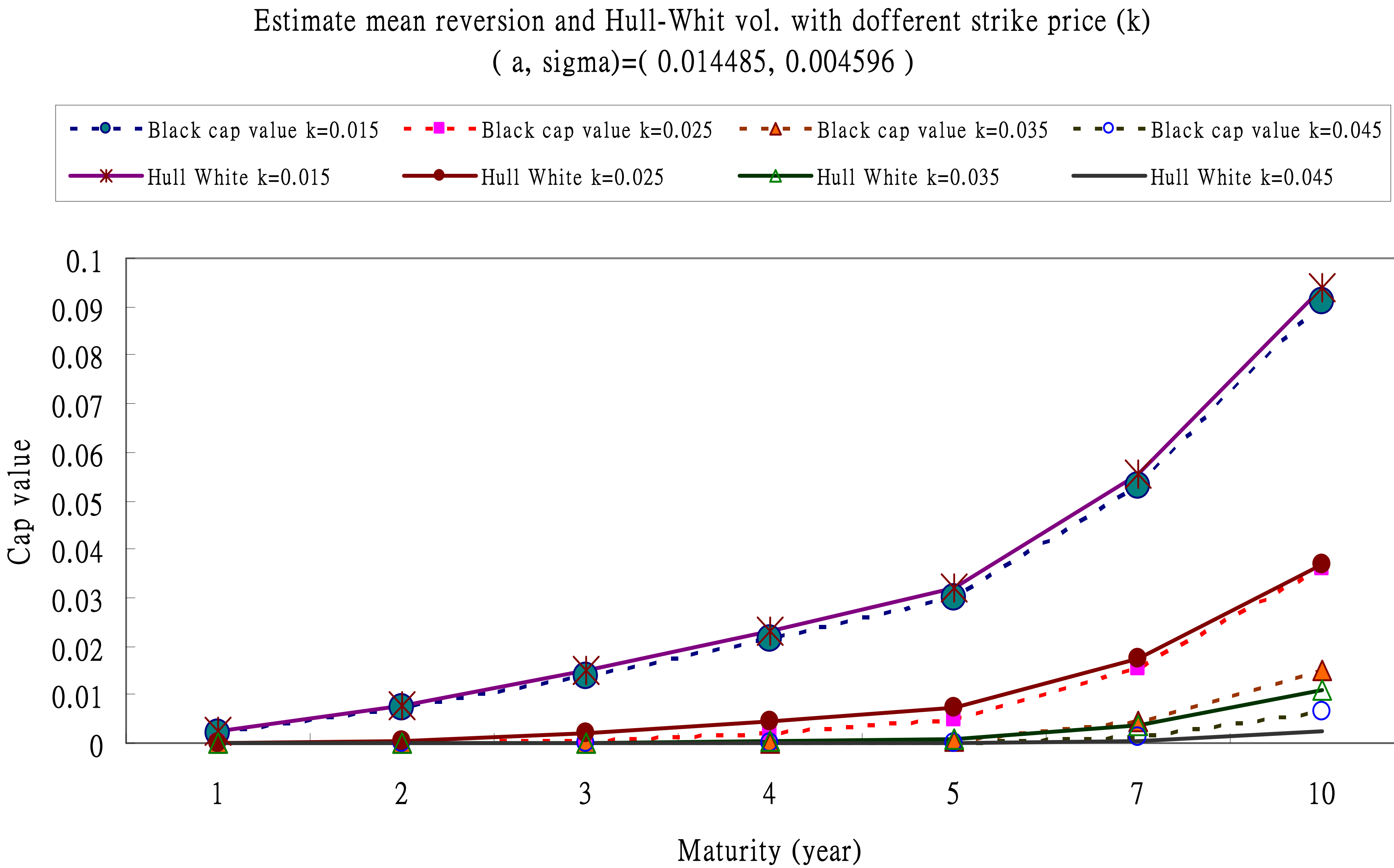
$$(1+K_{R\_cap}\Delta k)\max(K_{cap}-S,0)$$

$$S=\frac{1}{1+r_k\Delta k} \qquad K_{cap}=\frac{1}{1+K_{R\_cap}\Delta k}$$

- put option by Hull-White model :

$$KP(0,T)N(-h+\sigma_p)-LP(0,s)N(-h)$$

$$h=\frac{1}{\sigma_p}\ln\frac{LP(0,s)}{KP(0,T)}+\frac{\sigma_p}{2} \qquad \sigma_p=\frac{\sigma}{a}[1-e^{-a(s-T)}]\sqrt{\frac{1-e^{-2aT}}{2a}}$$



optimal estimated parameters : mean reversion= 0.014485, volatility= 0.004596
The minimum summation of square error (SSE) = 7.80133e-005

評價Snowball Notes
Par value =Market price=$1

Year	Coupon rate (Cn,i)
1	C1,i=3%, i=1,2,3,4
2	C2,i=C2,i-1+1.40%-FR2,i
3	C3,i=C3,i-1+1.65%-FR3,i
4	C4,i=C4,i-1+1.90%-FR4,i
5	C5,i=C5,i-1+2.15%-FR5,i
6	C6,i=C6,i-1+2.40%-FR6,i
7	C7,i=C7,i-1+2.65%-FR7,i
8	C8,i=C8,i-1+2.90%-FR8,i
9	C9,i=C9,i-1+3.15%-FR9,i
10	C10,i=C10,i-1+3.40%-FR10,i

Non-redeemable snowball price
1.13901
Redeemable snowball price
0.880214
Option value for redemptive article
0.258796

結　　論

- Snowball Notes is an inverse floating rate bond with freeze at zero and redemptive articles.
- Contribution: Provide an innovative polynomial-time pricing algorithm based on simple term structure model, Hull-White short rate model, to price snowball notes.
- Sensitivity analysis: Snowball price has negative relation to mean reversion and market rate, and positive relation to volatility, spreads and redemptive time.
- Issuers can effectively hedge by redemptive articles.

Thanks for your listening !