Bond Price Volatility

Financial Engineering and Computations

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Outline

- Price Volatility
- Duration
- Convexity
- Immunization

Price Volatility

- Price volatility measures the sensitivity of the percentage price change to changes in interest rates (interest rate risk).
- It is key to the risk management of interest-rate-sensitive securities.
- Define price volatility by

$$-\frac{\partial P / P}{\partial y} \longrightarrow \text{It is also so-call modified duration!}$$

$$\frac{\partial P}{P}(\textit{percent price change}) \approx -D \times \partial y$$

Numerical Example: Percentage Change of Bond Price

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1 %, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

General speaking, the duration we talk about is modified duration!

Behavior of Price Volatility

- Price volatility increases as the coupon rate decreases.
  - Zero-coupon bonds are the most volatile.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility

- For bonds selling above par or at par, price volatility increases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.

  (But the *yields* of long-term bonds are less volatile than those of short-term bonds.)
- For bonds selling below par, price volatility first increases then decreases.
  - Longer maturity here cannot be equated with higher price volatility.

Figure 4.1 **(Premium bonds and par bonds):**
Volatility with respect to terms to maturity

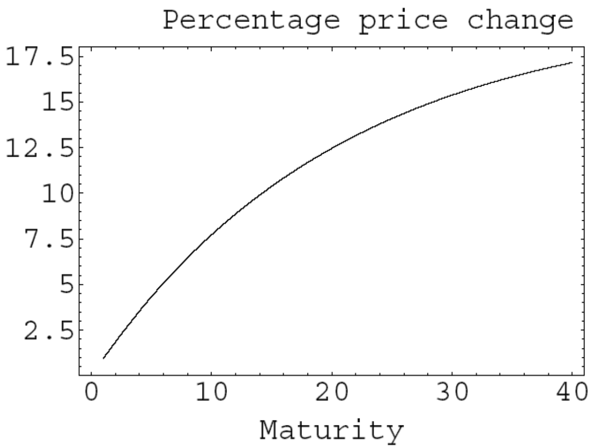
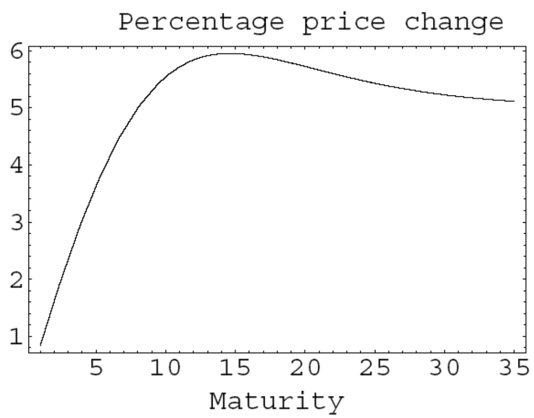


Figure 4.1 **(discount bonds):**
Volatility with respect to terms to maturity.



Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price. Formally,

$$MD \equiv \frac{1}{p}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

- The Macaulay duration, in periods, is equal to

$$MD = -(1+y)\frac{\partial P / P}{\partial y} \qquad (4.2)$$

The Proof

$$\because P = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C+F}{(1+y)^n}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{-C}{(1+y)^2} + \frac{-2C}{(1+y)^3} + \ldots + \frac{-n(C+F)}{(1+y)^{n+1}}$$

$$\therefore \frac{\partial P}{\partial y} = -\frac{1}{1+y}\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+R)^n}\right]$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}\left[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}\right]\frac{1}{P}$$

$$Define: MD = \frac{\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}}{P} = \frac{\sum_{i=1}^{n} iC_i}{P}$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}MD \Rightarrow \frac{\partial P / P}{\partial y / 1+y} = -MD$$

Example:

Duration of 6-Year Eurobond, 1,000 Face Value, 8 Percent Coupon and Market Yields 8%

t	C_t	DF_t	$C_t \times DF_t$	$C_t \times DF_t \times t$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1080	0.6302	680.58	4083.48
			1000	4992.71
MD=4992.71/1000=4.993 years				

C is cash flow, DF is discount factor

C++: Macaulay Duration的計算

- Macaulay Duration的計算

$$MD = \frac{1}{P}\left(\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}\right)$$

- 利用for loop同時求算 $\frac{1}{P}$ 和 $\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}$
- 相乘即爲答案

完整程式碼

```c
#include <stdio.h>
void main()
{
    int n;
    float c, r, Value=0,Discount,Duration=0;
    printf("請輸入期數:");
    scanf("%d",&n);
    printf("請輸入債息:");
    scanf("%f",&c);
    printf("請輸入利率:");
    scanf("%f",&r);
    for(int i=1;i<=n;i=i+1)
    {
     Discount=1;
     for(int j=1;j<=i;j++)
     {
      Discount=Discount/(1+r);
     }
     Duration=Duration+i*Discount*c;
     Value=Value+Discount*c;
     if(i==n)
     {
      Value=Value+Discount*100;
      Duration=Duration+n*Discount*100;
     }
    }
    Duration=Duration/Value;
    printf("Duration=%f",Duration);
}
```

For迴圈: 計算 Duration,Value

For迴圈: 計算Discount factor

If 條件式: i等於n時,考慮face value

Homework

- Program Exercise

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Macaulay Duration

- The MD of a coupon bond is less than its maturity.
- The MD of a zero-coupon bond
- The MD of a Consol

MD of a Coupon Bond

- The MD of a coupon is

$$MD = \frac{1}{P}\left(\sum_{i=1}^{n}\frac{iC}{(1+y)^i}+\frac{nF}{(1+y)^n}\right) \qquad (4.3)$$

Where C is the period fixed interest flow.

The MD of a zero-coupon bond

- MD of a zero-coupon bond is it's final maturity (n).
- Proof: because no cash flows before maturity, the MD is

$$MD = \frac{\sum_{i=1}^{n} iC_i(1+y)^{-i}}{\sum_{t=1}^{n} C_t(1+y)^{-i}} = \frac{nC_n(1+y)^{-n}}{C_n(1+y)^{-n}} = n$$

The MD of a Consol Bond

- A consol bond pay a fixed coupon each period but it never matures. $(Maturity\ \ date = \infty)$
- The duration of a consol bond is: $MD_c = 1 + \frac{1}{y}$

$$\because P = \frac{C}{y} \Rightarrow C = Py$$

$$\because MD = \frac{\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \frac{3C}{(1+y)^3} + ...\infty}{P} = \frac{\frac{Py}{(1+y)} + \frac{2Py}{(1+y)^2} + \frac{3Py}{(1+y)^3} + ...\infty}{P}$$

$$= \frac{y}{(1+y)} + \frac{2y}{(1+y)^2} + \frac{3y}{(1+y)^3} + ...\infty$$

$$\because \frac{1}{(1+y)} MD = \frac{y}{(1+y)^2} + \frac{2y}{(1+y)^3} + \frac{3y}{(1+y)^4} + ...\infty$$

$$\Rightarrow MD - \frac{1}{(1+y)} MD = \frac{y}{(1+y)} + \frac{y}{(1+y)^2} + \frac{y}{(1+y)^3} + ...\infty \Rightarrow \frac{y}{(1+y)} MD = \frac{y/(1+y)}{1 - \frac{1}{(1+y)}} = 1 \Rightarrow MD = \frac{1+y}{y} = 1 + \frac{1}{y}$$

Where y is yield to maturity

The MD of Floating-rate instruments

- A floating-rate instrument makes interest rate payments based on some publicized index such as the London Interbank Offered Rate (LIBOR), the U.S. T-bill rate.
- Instead of being locked into a number, the coupon rate is reset periodically to reflect the prevailing interest rate.
- Floating-rate instrument are typically less sensitive to interest rate changes than are fixed-rate instrument.

The MD of Floating-rate instruments

- Assume that the fixed coupon rate **c** in first *j* period, **y** in *n-j* period, also market yield is **y** now. The first reset date is ***j*** period from now, and reset will be performed thereafter.
- Let the principal be $1 for simplicity. The cash flow of the floating-rate instrument is

$$\underbrace{c, c, \ldots\ldots c}_{j},\ \underbrace{y, \ldots\ldots\ldots y, y+1}_{n-j}$$

- The MD of a floating-rate instrument is $MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^i}$

MD_{Fix}: Denote the MD of an otherwise identical fixed-rate bond.

Homework

- Prove that

$$MD_{floating} = MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}}$$

Where the bond is priced at par ,and the principal be $1 for simplicity.

Conversion

- To convert the MD to be year based, modify(4.3) as follow:

$$\frac{1}{p}\left(\sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1+\frac{y}{k})^{i}} + \frac{n}{k} \frac{F}{(1+\frac{y}{k})^{n}}\right)$$

Where ***y*** is the *annual yield* and ***k*** is the compounding frequency per annum.

- Equation (4.2) also becomes $MD = -(1+\frac{y}{k})\frac{\partial P / p}{\partial y}$
- Note from the definition that $MD(in\ years) = \frac{MD(in\ periods)}{k}$

Difference of formulas

- Macaulay Duration: $MD = -\frac{\partial P / P}{\partial y / (1+y)}$
- Modified Duration: $D = \frac{MD}{1+y} = -\frac{\partial P / P}{\partial y}$
- Dollar Duration: $DD = D \times P = -\frac{\partial P}{\partial y}$

Effective Duration

- A general numerical formula for volatility is the effective duration, defined as

$$\frac{P_{-} - P_{+}}{P_0 (y_{+} - y_{-})} \qquad (4.5)$$

where P_{-} is the price if the yield is decreased by Δy, P^{+} is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, and Δy is small.

- We can compute the effective duration of just about any financial instrument.

Effective Duration

- Most useful where yield changes alter the cash flow or securities whose cash flow is so complex that simple formulas are unavailable
- Duration of a security can be longer than its maturity or negative.

Immunization and MD

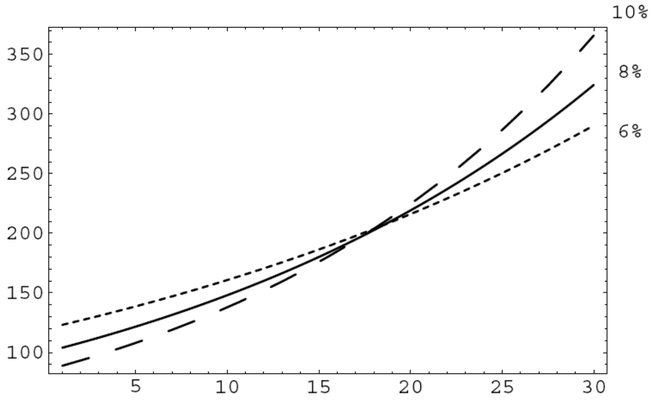
- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- How do we find such a bond portfolio?

→A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability.

– At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall.

– Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise

Bond value under three rate scenarios



Example: Immunization

- A $100,000 liability **12 years (maturity)** from now should be matched by a portfolio with a **MD of 12 years** and a FV of $100,000.

Immunization

- Assume the liability is ***L*** at time ***m*** and the current interest rate is ***y***. We are looking for a portfolio such that
  - **(1)** FV is *L* at the horizon *m*;
  - **(2)** $\partial FV/\partial y = 0$;
  - **(3)** FV is convex around *y*.
- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean ***L*** is the portfolio's minimum **FV** at horizon for small rate changes.

The Proof (1)

- Let $FV \equiv P(1+y)^m$, where P is the PV of the portfolio. Now,

$$\frac{\partial FV}{\partial y} = m(1+y)^{m-1}P + (1+y)^m \frac{\partial P}{\partial y} \qquad (4.8)$$

- Imposing Condition (2) leads to

$$m = -(1+y)\frac{\partial P/P}{\partial y} \qquad (4.9)$$

- The MD is equal to the horizon *m*.

The Proof (2)

- Employ coupon bond for immunization, because

$$FV = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}}$$

- It follows that

$$\frac{\partial^2 FV}{\partial^2 y} > 0,\ for\ y > -1 \qquad (4.10)$$

- Because the FV is convex for y>1, the minimum value of FV is indeed **L**.

Example: Immunization by using duration technique

- Suppose that we are in 2007, and the insurer has to make a guarantee payment **$1,469** to a policyholder in 5 years, 2012. The amount is equivalent to investing $1,000 at an annually compound rate of 8% over 5 years.
- Strategy1: Buy five-year maturity discount bonds.
- Strategy2: Buy five-year duration coupon bonds.

Strategy1: Buy five-year maturity discount bonds

- If the insurer buy 1.469 units of these bonds at a total cost of $1000 in 2007, these investment would produce exactly $1469 on maturity in five years.
- The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability.

$$P = \frac{1000}{1.08^5} = 680.58 \Rightarrow total\ cost = 1.469 \times 680.58 = 1000$$

$$cash\ flow\ in\ five\ years = \$1000 \times 1.469 = \$1469$$

Strategy2: Buy five-year duration coupon bonds.

- The gain or losses on reinvestment income that result from an interest rate change are exactly offset by losses or gains from the bond proceeds on sale.

	YTM fall to 7%	YTM is 8%	YTM rise to 9%
Coupons (5x$80)	400	400	400
Reinvestment income	60	69	78
Sale of bond at end of the 5th year	1009	1000	991
	$1469	$1469	$1469

Cash matching

Immunization

- If there is no single bond whose MD match the horizon, a portfolio of two bonds A and B, can be assembled by the solution of

$$1 = \omega_A + \omega_B$$
$$D = \omega_A D_A + \omega_B D_B \text{ (See next page)}$$

Here, $\mathbf{D_i}$ is the MD of bond i and $\boldsymbol{\omega_i}$ is the weight of bond i in the portfolio.

- Make sure that D falls between D_A and D_B to guarantee $\omega_A > 0, \omega_B > 0$, and positive portfolio convexity.

$$Set\ \ D = \frac{1}{P}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

$$D_A = \frac{1}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i}\ ,\ D_B = \frac{1}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

(A_i, B_i : cashflow of A and B at i-th period)

$$\therefore W_A D_A + W_B D_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$Set\ \ P = W_A P + W_B P$$

$$we\ can\ buy\ \frac{W_A P}{P_A}\ units\ of\ A,\ and\ \frac{W_B P}{P_B}\ units\ of\ B.$$

$$then\ D = \frac{1}{P}\sum_{i=1}^{n_A}\frac{i\frac{W_A P}{P_A}A_i}{(1+y)^i} + \frac{1}{P}\sum_{i=1}^{n_B}\frac{i\frac{W_B P}{P_B}B_i}{(1+y)^i} = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$\therefore\ D\ =\ W_A D_A + W_B D_B$$

In Class Exercise

- The liability has an MD of 3 years, but the money manager has access to only two kinds of bonds with MDs of 1 year and 4 years. What is the right proportion of each bond in the portfolio in order to match the liability's MD?

Limitations of Duration

- Duration matching can be costly.
- Immunization is a dynamic problem.
  - Because continuous rebalancing may not be easy to do and involves costly transaction fees.
  - There is a trade-off between being perfectly immunized and the transaction costs of maintaining.
- Large interest rate and convexity (see next figure).
  - Duration accurately measures the price sensitivity of fixed-income securities for small change in interest rates.

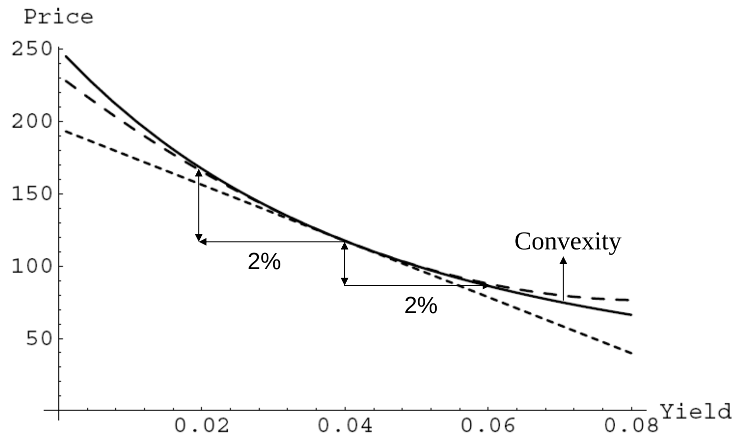
Convexity

- Convexity is defined as

$$convexity\,(in\ period) \equiv \frac{\partial^2 P}{\partial y^2}\frac{1}{P} \qquad (4.14)$$

- The convexity of a coupon bond is positive.
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Between two bonds with the same duration, the one with a higher convexity is more valuable.

Figure 4.6: Linear and quadratic approximation to bond price changes.



Convexity

- Convexity measured in periods and convexity measured in years are related by

$$convexity\,(in\ years) = \frac{convexity\,(in\ periods)}{k^2}$$

When there are k periods per annum.

- The convexity of a coupon bond increases as its coupon rate decreases. (next page)
- For a given yield and duration, the convexity decreases as the coupon decreases. (next page)

Properties of Convexity

CX Varies with Coupon (all variables may different)		**For same duration and yield, zero-Coupon bonds are less convexity**	
A	B	A	B
N=6	N=6	N=6	N=5
Y=8%	Y=8%	Y=8%	Y=8%
C=8%	C=0%	C=8%	C=0%
D=5	D=6	D=5	D=5
CX=28	CX=36	CX=28	CX=25.72

A and B are two different bond. C is coupon rate, Y is yield of maturity, D is duration, and CX is convexity.

Convexity

- The approximation $\Delta P/P \approx -$ (modified) duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 = -duration \times \Delta y + \frac{1}{2} \times convexity \times (\Delta y)^2$$

The Proof of Formula

$$dP = \frac{\partial P}{\partial y}dy + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(dy)^2 + \frac{1}{6}\frac{\partial^3 P}{\partial y^3}(dy)^3 + ...$$

$$\therefore \frac{dP}{P} = \frac{\partial P}{P}\frac{1}{\partial y}dy + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P}{\partial y^2}(dy)^2 + \frac{1}{6}\frac{1}{P}\frac{\partial^3 P}{\partial y^3}(dy)^3 + ...$$

$$\therefore \frac{dP}{P} = \frac{\partial P}{P}\frac{1+y}{\partial y}\frac{dy}{1+y} + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P}{\partial y^2}(dy)^2 + \frac{1}{6}\frac{1}{P}\frac{\partial^3 P}{\partial y^3}(dy)^3 + ...$$

$$\therefore \frac{dP}{P} = -MD\frac{dy}{1+y} + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P}{\partial y^2}(dy)^2 + \frac{1}{6}\frac{1}{P}\frac{\partial^3 P}{\partial y^3}(dy)^3 + ...$$

$$\therefore \frac{dP}{P} = -MD\frac{dy}{1+y} + \frac{1}{2}(CX)(dy)^2.....$$

$$\therefore \frac{dP}{P} = -MD_{\text{mod}}(dy) + \frac{1}{2}(CX)(dy)^2.....$$

$$\text{Define: CX(convexity)} = \frac{1}{\text{P}}\frac{\partial^2 \text{P}}{\partial \text{y}^2}, \text{ value of CX} = \frac{1}{2}\text{CX(dy)}^2$$

Calculation of Convexity

- Formula:

$$CX = \text{Scaling factor}(\text{The capital loss from 1bp rise} + \text{The capital gain from 1bp fall})$$
$$= 10^8\left(\frac{\Delta P^-}{P} + \frac{\Delta P^+}{P}\right)$$

- Example: To calculate convexity of the 8 percent coupon, 8 percent yield, six-year maturity Eurobond that had a price of $1000:

$$CX = 10^8\left(\frac{999.53785 - 1000}{1000} + \frac{1000.46243 - 1000}{1000}\right)$$
$$= 10^8(0.00000028) = 28$$

Example

- Given convexity C, the percentage price change expressed in percentage terms is approximated by $-D \times \Delta r + C \times (\Delta r)^2/2$ when the yield increases instantaneously by Δr%.
- For example, if $D = 10$, $C = 1.5$, and $\Delta r = 2\%$, price will drop by 17% because

$$\Delta P/P = -10 \times 2 + 1/2 \times 1.5 \times 2^2 = -17$$

In Class Exercise

- Show that the convexity of a zero-coupon bond is $n(n+1)/(1+y)^2$

Immunization (barbell Portfolio)

- Two bond portfolios with varying duration pairs D_A,D_B can be assembled to satisfy $D = \omega_A D_A + \omega_B D_B$ However, which one is to be preferred?
- Let there be n kinds of bonds, with bond i having duration D_i and convexity C_i, where $D_1 < D_2 < \ldots < D_n$. We then solve the follow constrained optimization problem:

$$\begin{aligned} \text{maximize} \quad & \omega_1 C_1 + \omega_2 C_2 + \ldots\ldots + \omega_n C_n \\ \text{subject to} \quad & \omega_1 + \omega_2 + \ldots\ldots \omega_n = 1 \\ & \omega_1 D_1 + \omega_2 D_2 + \ldots\ldots + \omega_n D_n = D \end{aligned}$$

The solution usually implies a barbell portfolio, which consists of very short-term bonds and very long-term bonds..

Lagrange Multiplier Method

$$function\ f(x_1, x_2, ..., x_n)$$
$$subject\ to\ g(x_1, x_2, ..., x_n) = 0$$
$$F(x_1, x_2, ..., x_n, \lambda) = f(x_1, x_2, ..., x_n) + \lambda \bullet g(x_1, x_2, ..., x_n)$$
$$Fx_1(x_1, x_2, ..., x_n, \lambda) = 0$$
$$Fx_2(x_1, x_2, ..., x_n, \lambda) = 0$$
$$\vdots$$
$$Fx_n(x_1, x_2, ..., x_n, \lambda) = 0$$
$$g(x_1, x_2, ..., x_n) = 0$$

A Simple Example

$$\min \quad f(x,y) = 5x^2 + 6y^2 - xy$$
$$s.t \quad x + 2y = 24$$

$$g(x,y) = x + 2y - 24 = 0$$

$$F(x,y,\lambda) = 5x^2 + 6y^2 - xy + \lambda(x + 2y - 24)$$
$$F_x(x,y,\lambda) = 10x - y + \lambda = 0....(1)$$
$$F_y(x,y,\lambda) = 12y - x + 2\lambda = 0..(2)$$
$$g = x + 2y - 24 = 0..................(3)$$

根據(1)、(2)　得$x = \frac{2}{3}y$

$x = \frac{2}{3}y$　代入(3), 得$y = 9, x = 6$

Use Lagrange Multiplier Method to obtain the optimal bond portfolio

$$\max \ \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n$$
$$s.t \quad \omega_1 + \omega_2 + \ldots + \omega_n = 1$$
$$\omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n = D$$
$$g_1(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 + \omega_2 + \ldots + \omega_n - 1 = 0$$
$$g_2(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D = 0$$
$$F(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n + \lambda_1(\omega_1 + \omega_2 + \ldots + \omega_n - 1) + \lambda_2(\omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D)$$

Example: Immunization (Convexity is desirable)

- Consider a pension fund manger with a 15-year payout horizon. To immunize the risk of interest rate changes, the manger purchase bonds with a 15-year duration. Consider two alternative strategies to achieve this:

- Strategy1: Invest 100 percent of resources in a 15-year deep-discount bond with an 8 percent yield. (Bullet portfolio)
- Strategy2: Invest 50 percent in the very short-term money market and 50 percent in 30-year deep-discount bond with an 8 percent yield. (Barbell portfolio)

Example: Immunization (Convexity is desirable)

- Strategy1:
  Duration =15
  Convexity =206
  value of the convexity $=1/2 \times \text{convexity} \times \triangle y^2 = 25.75\%$

  $\triangle y = 5\%$
- Strategy2:
  Duration $=1/2 \times 0 + 1/2 \times 30 = 15$
  Convexity $=1/2 \times 0 + 1/2 \times 797 = 398.5$
  Value of the convexity $=1/2 \times \text{convexity} \times \triangle y^2 = 49.81\%$

  High convexity is more valuable
- The manger may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as a result, both positive and negative shocks to interest rates would have beneficial effects on the net worth.

Categories of Immunization

- Cash matching
- Rebalancing

Cash matching

- Cash matching is the approach that a stream of liability can always be immunized with a matching stream of zero-coupon bonds.
- Two problem with this approach are that (1) zero-coupon bonds may be missing for certain matruity.(2) they typically carry lower yield.
- Recall example (Immunization by using duration technique).

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.
- The MD decreases as time passes.
- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  - Consider a coupon bond whose MD matches horizon.
  - Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.
  - So immunization needs to be reestablished even if interest rates never change.

Hedging

- Hedging aims to offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$DD \equiv modified\ duration \times price(\%\ of\ \ par) = -\frac{\partial P}{\partial y}$$

- The approximate dollar price change per \$100 of par value is

$$price\ change \approx -dollar\ duration \times yield\ change$$

Hedging

- Because securities may react to interest rate changes differently, we define yield beta to measure relative yield changes.

$$yield\ beta \equiv \frac{change\ in\ yield\ \ for\ the\ hedged\ security}{change\ in\ yield\ \ for\ the\ hedging\ security}$$

- Let the hedge ratio be

$$h \equiv \frac{dollar\ duration\ of\ \ the\ hedged\ security}{dollar\ duration\ of\ \ the\ hedging\ security} \times yield\ beta \qquad (4.13)$$

- Then hedging is accomplished when the value of the hedging security is h times that of the hedged security.

Example 4.2.2

- Suppose we want to hedge bond A with a duration of seven by using bond B with a duration of eight. Under the assumption that yield beta is one and both bonds are selling at par, the hedge ratio is 7/8, This means that an investor who is long \$1 million of bond A should short \$7/8 million of bond B.

Homework

- Start with a bond whose PV is equal to the PV of a future liability and whose MD exceeds the horizon. Show that the bond will fall short of the liability if interest rates rise and more than meet the goal if interest rates fall. The reverse is true the MD falls short of the horizon.