Term Structure of Interest Rates

Financial Engineering and Computations

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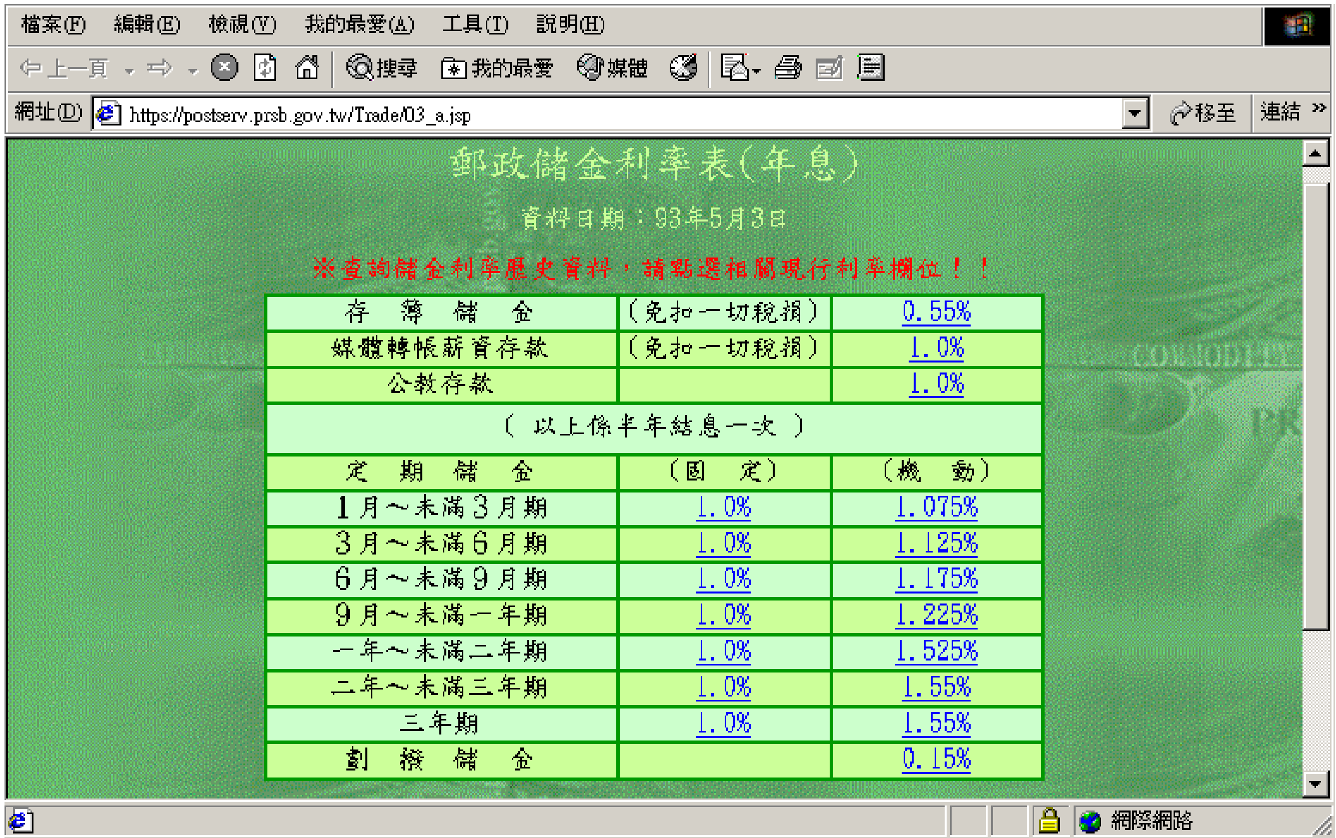
Outline

- Introduction
- Spot Rates
- Extracting Spot Rates from Yield Curves
- Spot Rate Curve , Forward Rate Curve, Yield Curve
- Forward Rates
- Locking in the Forward Rates
- Term Structure Theory

Term Structure of Interest Rates

- The interest rates vary with maturity.



郵政儲金利率表(年息)

資料日期：93年5月3日

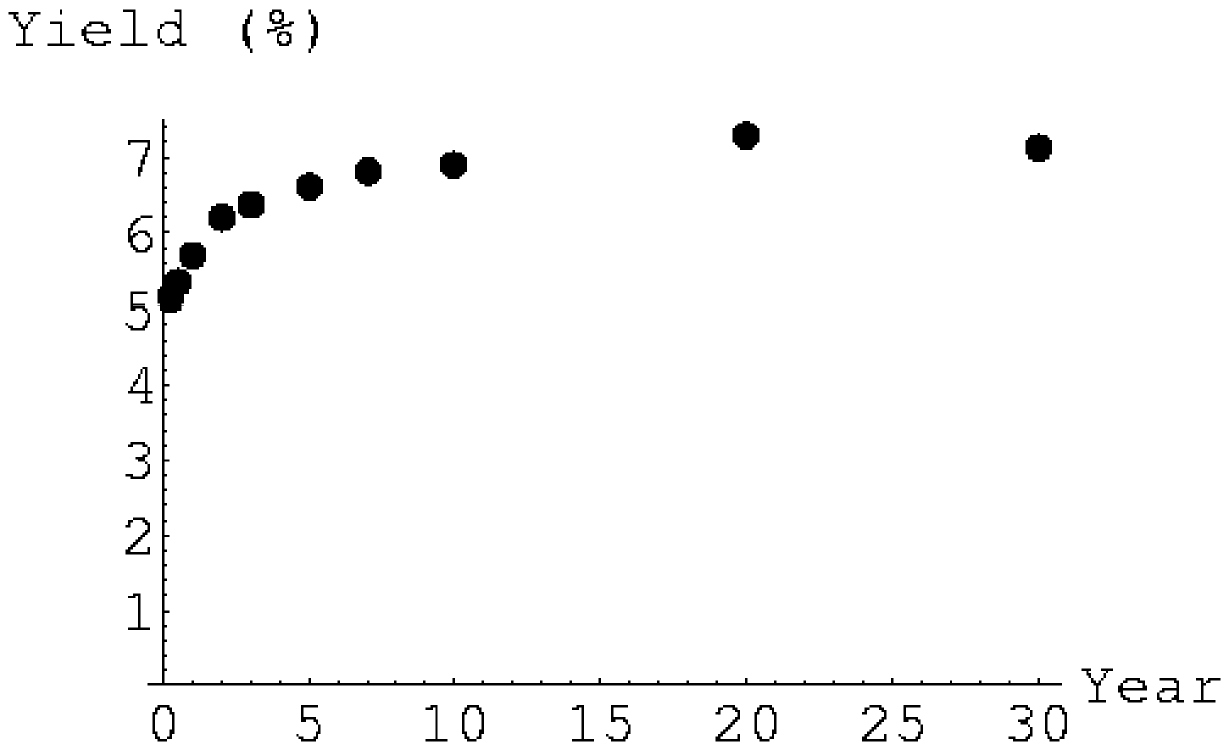
※查詢儲金利率歷史資料，請點選相關現行利率欄位！！

存簿儲金	(免扣一切稅捐)	0.55%
媒體轉帳薪資存款	(免扣一切稅捐)	1.0%
公教存款		1.0%
(以上係半年結息一次)		
定期儲金	(固定)	(機動)
1月～未滿3月期	1.0%	1.075%
3月～未滿6月期	1.0%	1.125%
6月～未滿9月期	1.0%	1.175%
9月～未滿一年期	1.0%	1.225%
一年～未滿二年期	1.0%	1.525%
二年～未滿三年期	1.0%	1.55%
三年期	1.0%	1.55%
劃撥儲金		0.15%

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
    - Credit spread.
  - They differ solely in their terms to maturity.

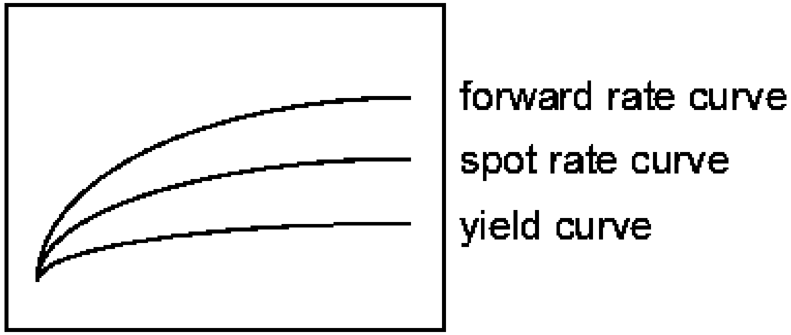
Yield Curve



Four Shapes

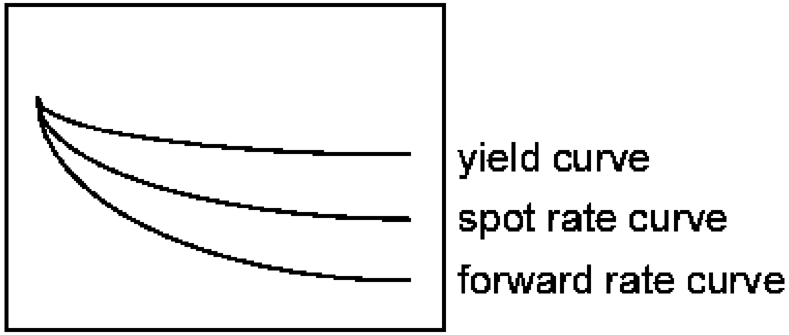
- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Normal curve



(a)

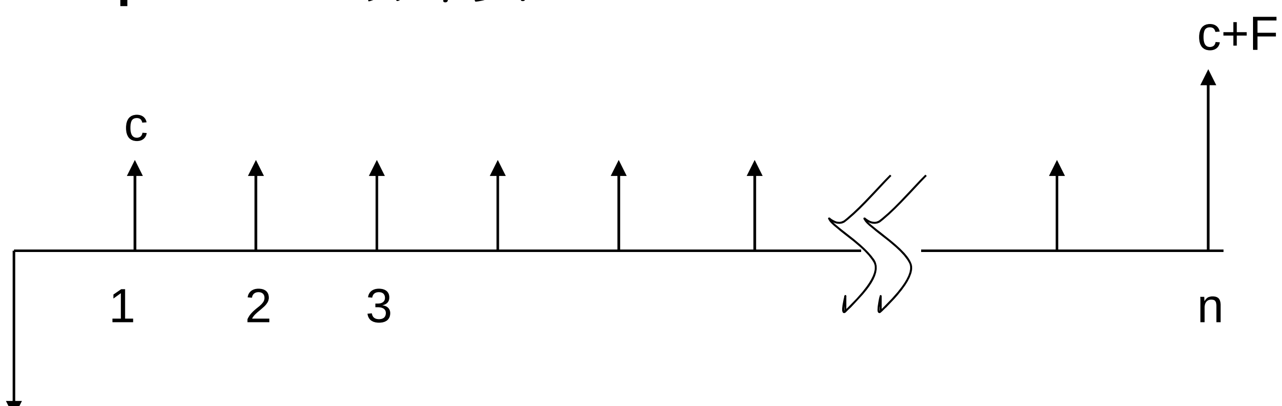
Inverted curve



(b)

Bond yield 和 Zero Rate

- Bond price 的計算

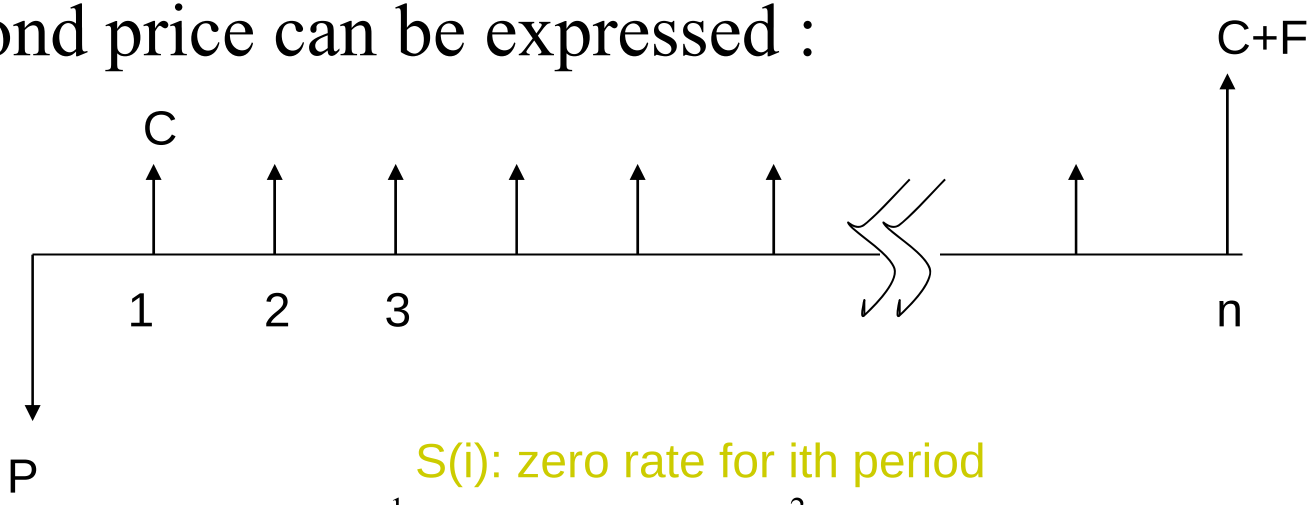


$$x = c \times (1+y)^{-1} + c \times (1+y)^{-2} + \ldots + (c+F) \times (1+y)^{-n}$$

- 不同的時間現金流用相同的y折現
  - y爲市場交易的價格
  - 不同的現金流結構無法用來折現
  - 使用zero rate 替代
- n-year zero rate (zero coupon rate): 可視爲n年後到期的零息債券的yield rate

Spot Rate (Zero rate)

- It is defined as yield of zero coupon bond
- A spot rate curve (zero-coupon yield curve) is a plot of spot rates against maturity.
- Bond price can be expressed :



S(i): zero rate for ith period

$$P = C \times (1+S(1))^{-1} + C \times (1+S(2))^{-2} + \ldots + (C+F) \times (1+S(n))^{-n}$$

y: bond yield

$$P = C \times (1+y)^{-1} + C \times (1+y)^{-2} + \ldots + C \times (1+y)^{-n} + F \times (1+y)^{-n}$$

市場上無法直接觀察長天期Spot rate

- 市場上交易的債券中
  - 短天期的債券多為zero coupon bond
  - 長天期的皆為coupon bearing bond
- 無法觀察長天期的Spot rate
- 可是市場折現利率應和maturity相關,
  - 求長天期的Spot rate實屬必須
- 藉由coupon bearing bond的拆解可得

方法簡介

- 考慮一個二期的模型,假定有兩個債券 B1,B2到期日分別爲第一和第二期. Bond yield分別爲 y1,y2,債券價格如下

$$P_1 = \frac{c+F}{(1+y1)} \qquad\qquad P_2 = \frac{c}{(1+y2)} + \frac{c+F}{(1+y2)^2}$$

- 考慮在第一期到期的零息債券,其zero rate S(1)=y1
- 考慮在第二期到期的零息債券,其zero rate S(2)計算如下:
  - B2可拆解成兩個零息債券,
    - 一個在第一期到期,面值爲c =>用 S(1)折現
    - 另一個在第二期到期,面值爲c+F =>用S(2)折現
  - 可得以下算式:

$$P2 = \frac{c}{(1+S(1))} + \frac{c+F}{(1+S(2))^2}$$

因S(1)已知,可求出S(2)

n期的Zero Rate計算

- 假定S(1), S(2), S(3),… S(n-1) 皆已知
- S(n)滿足以下式子:

$$Pn = \frac{c}{(1+S(1))} + \frac{c}{(1+S(2))^2} + \ldots + \frac{c+F}{(1+S(n))^n}$$

- 移項可得 $(1+S(n))^n = \dfrac{c+F}{Pn - \dfrac{c}{(1+S(1))} - \dfrac{c}{(1+S(2))^2} - \ldots - \dfrac{c}{(1+S(n-1))^{n-1}}}$

$$Sn = \sqrt[n]{\frac{c+F}{Pn - \frac{c}{(1+S(1))} - \frac{c}{(1+S(2))^2} - \ldots - \frac{c}{(1+S(n-1))^{n-1}}}} - 1$$

The procedure is called bootstrapping.

```
float ZeroRate[5];
float Yield[5];
float C;
scanf("%f",&C);
for(int i=0;i<5;i=i+1)
{
 printf("輸入Yield rate %d:",i+1);
 scanf("%f",&Yield[i]);
}
ZeroRate[0]=Yield[0];
for(i=1;i<=4;i++)
{
 float BondValue=0;
 for(int j=0;j<=i;j=j+1)
 {
        float Discount=1;
        for(int k=0;k<=j;k++)
        {
         Discount=Discount/(1+Yield[i]);

        }
   BondValue=BondValue+Discount*C;
        if(j==i)
        {
         BondValue=BondValue+Discount*100;
        }
 }
```

程式宣告

輸入殖利率

第一期Zero rate=Yield

計算第i+1期zero rate

計算債券價格Pi+1

```
for(j=0;j<i;j=j+1)
 {
   float PV=C;
        for(int k=0;k<=j;k++)
        {
         PV=PV/(1+ZeroRate[j]);
        }
   BondValue=BondValue-PV;
 }
 ZeroRate[i]=pow((C+100)/BondValue,1.0/(i+1))-1
}
```

$$Bn-\frac{c}{(1+Z1)}-\frac{c}{(1+Z2)^2}-\dots-\frac{c}{(1+Zn-1)^{n-1}}$$

```
for(i=0;i<=4;i++)
{
 printf("第%d期zero rate=%f\n",i,ZeroRate[i]);
}
}
```

列印zero rate

見 第四章 ZeroCurve project

Compare Yield Curve and Spot rate curve

- Spot rate curve is zero coupon yield curve or zero curve.
- Spot rate curve is consisted of zero rate.
  - zero coupon bond
- Yield curve is consisted of bond yield.
  - coupon bearing bond、zero coupon bond

Example

- Suppose the 1-year T-bill has yield of 8%. Because this security is a zero-coupon bond, the 1-year spot rate is 8%. When the 2-year 10% T-note is trading at 90, the 2-year spot rate satisfies

$$\because 90 = \frac{10}{1.08} + \frac{110}{(1+S(2))^2} \Rightarrow S(2) = 0.1672 \; or \; 16.72\%$$

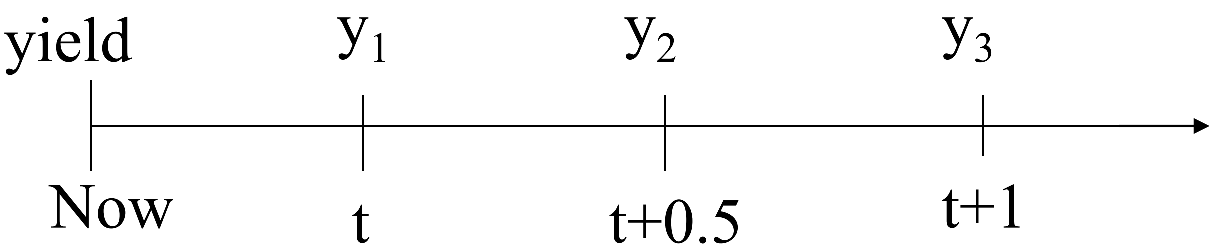
Extracting Spot Rates from Yield Curve
(continuous compounding)

- Suppose three bonds paid semiannually with yield **y** and interest **C**.
- The prices of bonds :

$$P_1 = (C+F)\times e^{-y_1\times t}$$

$$P_2 = C\times e^{-y_2\times t} + (C+F)\times e^{-y_2\times(t+0.5)}$$

$$P_3 = C\times e^{-y_3\times t} + C\times e^{-y_3\times(t+0.5)} + (C+F)\times e^{-y_3\times(t+1)}$$



Extracting Spot Rates from Yield Curve
(continuous compounding)

- Let three kinds of zero rate is **S(1),S(2),S(3).**
- Given **y_1=S(1).**
- We can obtain the spot rate from process as below.

$$P_1 = (C+F)e^{-y_1 \times t} = (C+F)e^{-S(1)\times t}$$

$$P_2 = Ce^{-y_2 \times t} + (C+F)e^{-y_2 \times (t+0.5)}$$
$$= Ce^{-S(1)\times t} + (C+F)e^{-S(2)\times(t+0.5)} \longrightarrow \text{We can get S(2)}$$

$$P_3 = Ce^{-y_3 \times t} + Ce^{-y_3 \times (t+0.5)} + (C+F)e^{-y_3 \times (t+1)}$$
$$= Ce^{-S(1)\times t} + Ce^{-S(2)\times(t+0.5)} + (C+F)e^{-S(3)\times(t+1)} \longrightarrow \text{We can get S(3)}$$

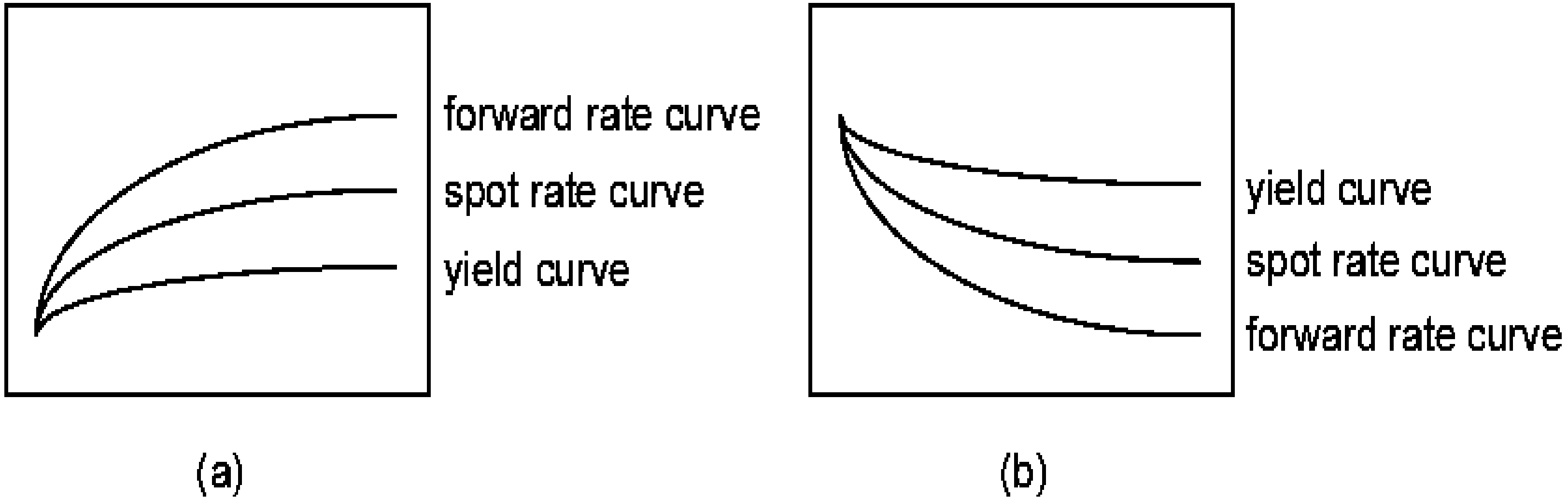
Class Exercise

- Suppose the 1-year T-bill has yield of 8%. Because this security is a zero-coupon bond, the 1-year spot rate is 8%. When the 2-year 10% T-note is trading at 90, please use continuous compounding to extract the 2-year spot rate.

Spot Rate Curve and Yield Curve

- If the yield curve is flat, the spot rate curve coincides with the yield curve.
- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$, if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$, if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$, if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$, if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

Figure 5.6: Shapes of Curves



Forward rate curves will be discussed later.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted, then the yield curve is inverted, and vice versa.
  - However, a normal yield curve does not guarantee a normal spot rate curve.
- When the final principal payment is relatively insignificant, the spot rate curve and the yield curve do share the same shape.

Shapes

- Consider a 3-period coupon bond that pays \$1 per period and repays the principal of \$100 at maturity.
- Assume spot rates $S(1) = 0.1$, $S(2) = 0.9$, and $S(3) = 0.901$.
- Yields to maturity are $y_1 = 0.1$, $y_2 = 0.8873$, and $y_3 = 0.8851$, not strictly increasing!

Homework

- 證明課本(C++財務程式設計)第四章習題 7(a), (b), (c)

Yield Spread

- Yield spread is the difference between the IRR of the risky bond and that of a risk-free bond with comparable same maturity.

$$P_{risk-free} = \sum_{i=1}^{n} C(1+y)^{-i} + F \times (1+y)^{-n}$$

$$P_{risky} = \sum_{i=1}^{n} c \times (1+y+y')^{-i} + F \times (1+y+y')^{-n}$$

Where y' is the yield spread.

Static Spread

- The static spread is the amount ***s*** by which the spot rate curve has to shift in parallel in order to price the risky bond correctly,

$$P_{risky} = \sum_{t=1}^{n} \frac{C_t}{\left(1 + s + S(t)\right)^t}$$

- Unlike the yield spread, the static spread incorporates information from the zero rate structure.
- The amount of static spread can be considered as the **constant credit spread** to the Treasury spot rate curve that reflects the risk premium of a corporate bond.

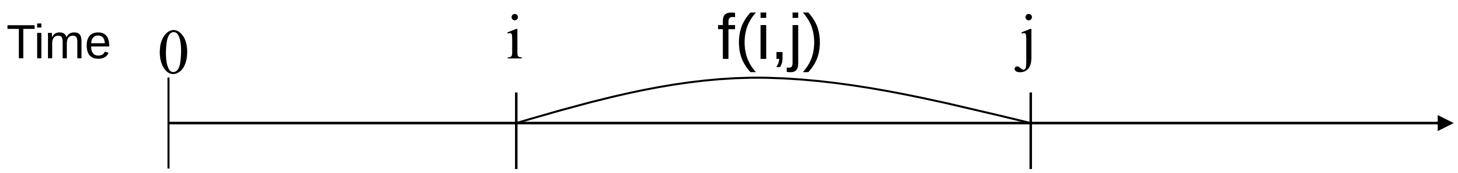
Homework

- 課本(C++財務程式設計)第四章習題5,6

Forward Rate

- The forward rate reflect information regarding future interest rates implied by the market.
- If we invest $1 from now to jth period.

$$(1+S(j))^j = (1+S(i))^i(1+f(i,j))^{j-i} \Rightarrow f(i,j) = \left(\frac{(1+S(j))^j}{(1+S(i))^i}\right)^{i-j} - 1$$



Example: Spot and Forward Rate

- In this example, if \$1 is invest in 5-period zero-coupon bond (maturity strategy), it will grow to be

$$\$1\times(1+0.04)^5=\$1.22$$

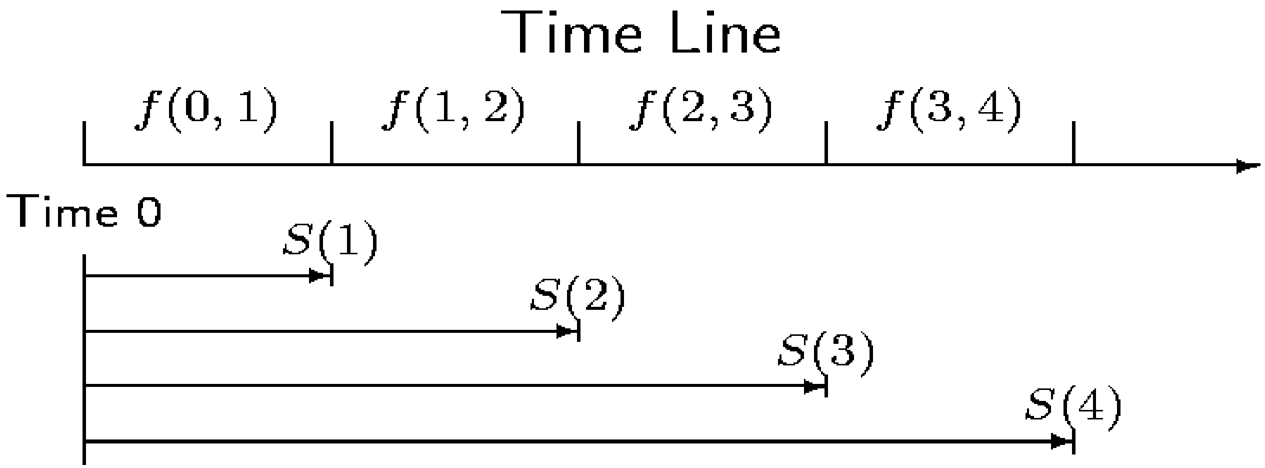
- An alternative strategy is to invest \$1 in one-period zero-coupon bonds at 2% and then reinvest at the one-period forward rates (rollover strategy). The result is exactly the same as expected.

$$(\$1\times1.02)\times1.03\times1.04\times1.05\times1.06=\$1.22$$

Forward rate

Forward Rates

- By definition, $f(0, j) = S(j)$.
- $f(i, j)$ is called the (implied) forward rates.
  - More precisely, the *(j-i)-period* forward rate i periods from now.



Forward Rate and Future Zero Rate

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- Term structure theories have different explanation.
  - Unbiased expectation theory.
    - $f(i,j)=E(S(i,j))$
  - Liquidity preference theory.
    - $f(i,j) > E(S(i,j))$

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\ S(a, b)\ ]$$

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$.
- Therefore, $E[\ S(j, j+1)\ ] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.

- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

Liquidity Preference Theory

- The liquidity preference holds that investors demand a risk premium for holding long-term bonds.
- This implies that $f(a,b) > E(S(a,b))$.
- Even if people expect the interest rate to decline and rise equally, the theory asserts that the curve is upward sloping more often.

Spot and Forward Rate under Continuous Compounding

- The formula for the forward rate:

$$\because e^{-j\times S(j)} = e^{-i\times S(i)} e^{-(j-i)\times f(i,j)}$$

$$\Rightarrow -jS(j) = -iS(i) - (j-i) f(i,j) \Rightarrow f(i,j) = \frac{jS(j) - iS(i)}{j-i}$$

- The spot rate is an arithmetic average of forward rates.

$$\because e^{-j\times S(j)} = e^{-S(1)} e^{-f(1,2)} e^{-f(2,3)} \ldots e^{-f(j-1,j)}$$

$$\Rightarrow -jS(j) = -S(1) - f(1,2) - f(2,3) \ldots - f(j-1,j)$$

$$\Rightarrow S(j) = \frac{f(0,1) + f(1,2) + f(2,3) \ldots + f(j-1,j)}{j}$$

Spot and Forward Rate under Continuous Compounding

- The one-period forward rate:

$$f(j, j+1) = (j+1)S(j+1) - jS(j) \qquad (5.10)$$

- Under continuous time instead of discrete time, the instantaneous forward rate at T time equals

$$\because f(T, T+\Delta T) = S(T+\Delta T) + \big(S(T+\Delta T) - S(T)\big)\frac{T}{\Delta T}$$

$$\Rightarrow f(T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) = S(T) + T\frac{\partial S}{\partial T} \qquad (5.11)$$

Note that f(T) $>$ S(T) if and only if $(\partial S/\partial T) > 0$

Example: Spot and Forward Rate

- Compute the one-period forward rates from this spot rate curve:

S(1):2.0%, S(2):2.5%, S(3):3.0%, S(4):3.5%, S(5):4.0%.

- Answer:

$$\because \frac{2+f(1,2)}{2}=2.5 \Rightarrow f(1,2)=3\%$$

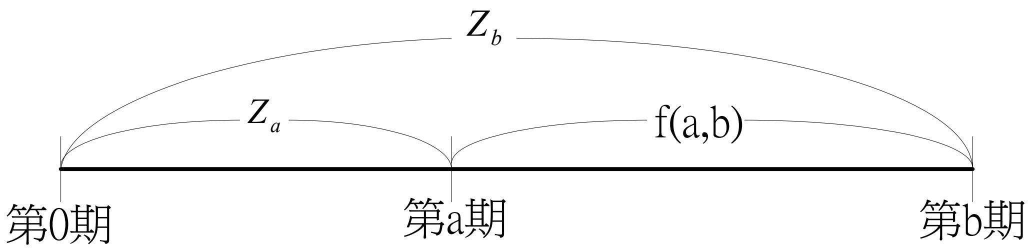
$$\because \frac{2+3+f(2,3)}{3}=3 \Rightarrow f(2,3)=4\%$$

$$\because \frac{2+3+4+f(3,4)}{4}=3.5 \Rightarrow f(3,4)=5\%$$

$$\because \frac{2+3+4+5+f(4,5)}{5}=4 \Rightarrow f(4,5)=6\%$$

C++:使用零息利率計算遠期利率

- 今天市場的零息利率期限結構，推論未來某段期間隱含的零息利率。



$$(1+Z_b)^b = (1+Z_a)^a (1+f(a,b))^{b-a}$$

$$f(a,b) = \sqrt[b-a]{\frac{(1+Z_b)^b}{(1+Z_a)^a}} - 1$$

使用二維陣列儲存遠期利率

行編號 列編號	0	1	2	3	4	5
0	f(0,0)	f(0,1)	f(0,2)	f(0,3)	f(0,4)	f(0,5)
1	X	f(1,1)	f(1,2)	f(1,3)	f(1,4)	f(1,5)
2	X	X	f(2,2)	f(2,3)	f(2,4)	f(2,5)
3	X	X	X	f(3,3)	f(3,4)	f(3,5)
4	X	X	X	X	f(4,4)	f(4,5)
5	X	X	X	X	X	f(5,5)

```
for(int i=0;i<5;i=i+1)
{
 printf("輸入第 %d期的零息利率:",i+1);
 scanf("%lf",&ZeroRate[i]);
}
```

輸入零息利率

```
for(i=0;i<5;i=i+1)
{
 Forward[0][i+1]=ZeroRate[i];
}
for(i=0;i<=5;i=i+1)
{
 Forward[i][i]=0;
}
```

ForwardRate[0,i]=ZeroRate[i]

ForwardRate[i,i]=0

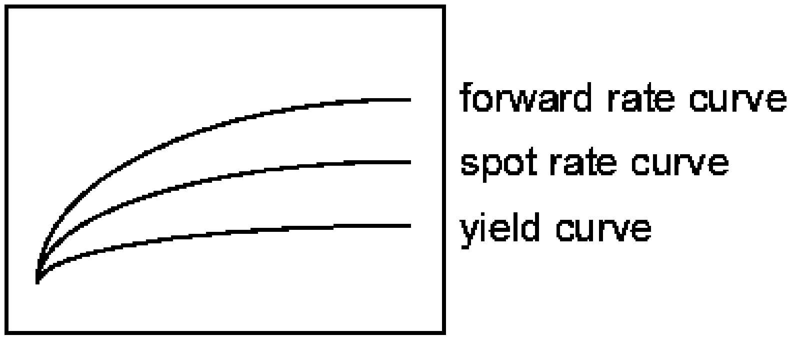
```
for(int a=1;a<5;a=a+1)//計算遠期利率
{
 for(int b=a+1;b<=5;b=b+1)
 {

Forward[a][b]=pow(pow(1+Forward[0][b],b)/pow(1+Forward[0][a],a),1.0/(b-a))-1;
 }
}
```

$$f(a,b) = \sqrt[b-a]{\frac{(1+Z_b)^b}{(1+Z_a)^a}} - 1$$

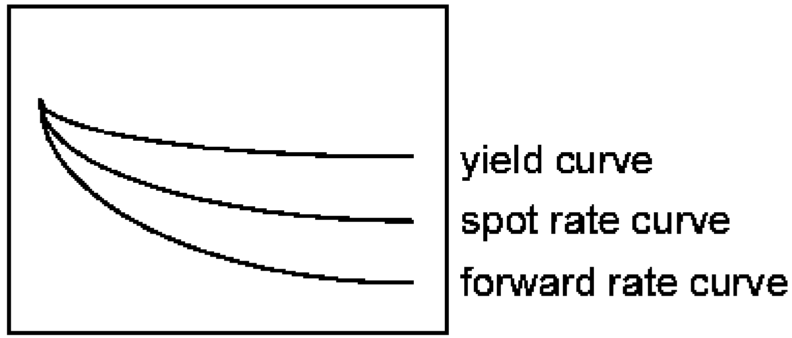
殖利率曲線，零息利率曲線，遠期利率曲線關係

Normal curve



(a)

Inverted curve



(b)

$$(1+Z_b)^b = (1+Z_a)^a (1+f(a,b))^{b-a}$$

當Zb>Za=> f(a,b)>Zb>Za

使用上述程式驗證零息利率和遠期利率關係

Spot Rate and Forward Rate

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$

Locking in the Forward Rates

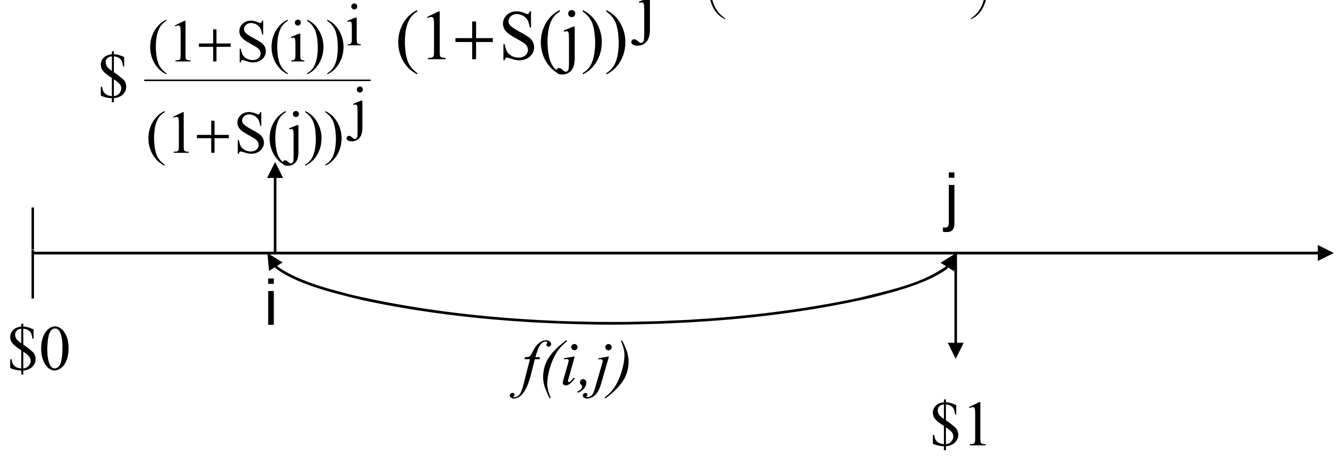
- Forward rates may not be realizes in the future ($f(i,j) \neq S(i,j)$), but we can lock in any forward rate $f(i,j)$.
- Now we can make following strategies.
  - Buy **1** unit *j-year* zero-coupon bond.
  - Sell $\frac{(1+S(i))^i}{(1+S(j))^j}$ units *i-year* zero-coupon bonds.
- No net initial investment, because

$$\frac{(1+S(i))^i}{(1+S(j))^j} \times \frac{1}{(1+S(i))^i} - 1 \times \frac{1}{(1+S(j))^j} = 0$$

Locking in the Forward Rates

- At time *j* there will be a cash inflow of \$**1**.
- At time *i* there will be a cash outflow of \$ $\frac{(1+S(i))^i}{(1+S(j))^j}$
- The cash flow stream implies the rate *f(i,j)* between times *i* and *j*.

$$\frac{(1+S(i))^i}{(1+S(j))^j} \times \left(1+f(i,j)\right)^{j-i} = 1$$



C++:遠期利率的鎖定

- 時間j時可拿到$1
- 時間i時需支付$ $\frac{(1+Zi)^i}{(1+Zj)^j}$
- 利率計算=>

$$\frac{(1+Zi)^i}{(1+Zj)^j}\times(1+f(i,j))^{j-i}=1$$

計算購買一單位的時間j到期的零息債券,須賣出多少單位時間i到期的零息債券,才能鎖住 遠期利率 f(i,j),其中0<i<j<=5,並將結果存在二維陣列LockStrategy[i,j]中

LockStratrgy 陣列

行編號 列編號	0	1	2	3	4	5
0	X	X	X	X	X	X
1	X	X	$\frac{(1+Z_1)^1}{(1+Z_2)^2}$	$\frac{(1+Z_1)^1}{(1+Z_3)^3}$	$\frac{(1+Z_1)^1}{(1+Z_4)^4}$	$\frac{(1+Z_1)^1}{(1+Z_5)^5}$
2	X	X	X	$\frac{(1+Z_2)^2}{(1+Z_3)^3}$	$\frac{(1+Z_2)^2}{(1+Z_4)^4}$	$\frac{(1+Z_2)^2}{(1+Z_5)^5}$
3	X	X	X	X	$\frac{(1+Z_3)^3}{(1+Z_4)^4}$	$\frac{(1+Z_3)^3}{(1+Z_5)^5}$
4	X	X	X	X	X	$\frac{(1+Z_4)^4}{(1+Z_5)^5}$
5	X	X	X	X	X	X

完整程式碼

```
#include <math.h>
#include <stdio.h>

void main()
{
  double unit_short[6][6];
  double ZeroRate[6];
  int i,j;
  //輸入零息利率
  for(i=1;i<=6;i++)
  {
        printf("請輸入第 %d期的零息利率: ",i);
        scanf("%lf",&ZeroRate[i-1]);
  }
  //計算鎖定f(i,j)需放空零息債券的單位數
  for( i=0;i<6;i++)
        for(j=i;j<6;j++)
        unit_short[i][j] = pow(1+ZeroRate[i],i)/pow(1+ZeroRate[j],j);
  //輸出結果
  for( i=0;i<5;i++)
        for(j=i+1;j<6;j++)
        printf("鎖定 f(%d,%d) -- 放空 %lf 單位 第%d期零息債券,買入一單位第%d期的零息
              債券\n",i+1,j+1,unit_short[i][j],i+1,j+1);
}
```

Homework

- Exercise

  The fact that forward rate can be locked in today means that future spot rates must equal today's forward rates, or S(a,b)=f(a,b), in a certain economy. Why? How about an uncertain economy? (Hint:可舉一個簡單實例，用套利的觀念來說明)