Option Basic

Financial Engineering and Computations

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Outline

- Introduction of Option
- Theory of Rational Option Pricing
- Put-Call Parity
- Option strategies

選擇權簡介(1)

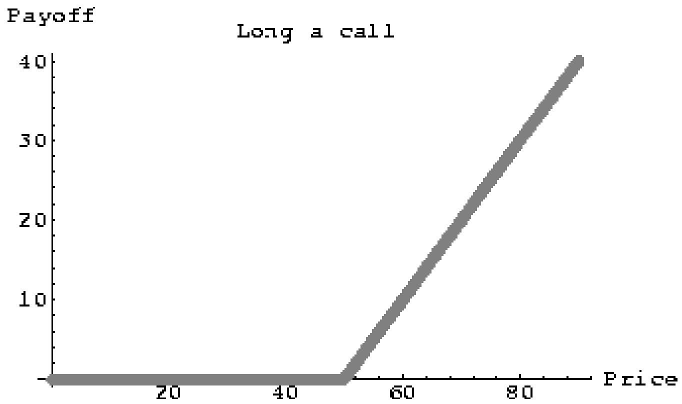
- 選擇權給予持有人在特定的時間點上, 以約定好的價格, 買入或賣出特定的資產的權利
  - 契約上載明的日期稱為到期日(maturity date)
    - 假定合約起始點為0, 到期日為T
  - 交易的資產稱為標的資產(underlying asset)
    - 假定在時間t時, 標的資產價格為S(t)
  - 契約上載明的價格稱為履約價格(exercise price)
    - 假定履約價格為X

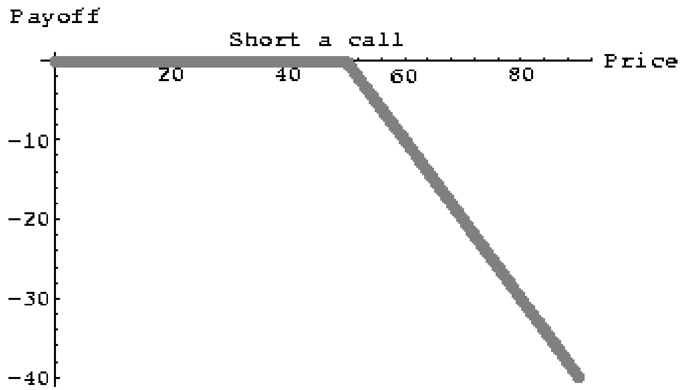
選擇權簡介(2)

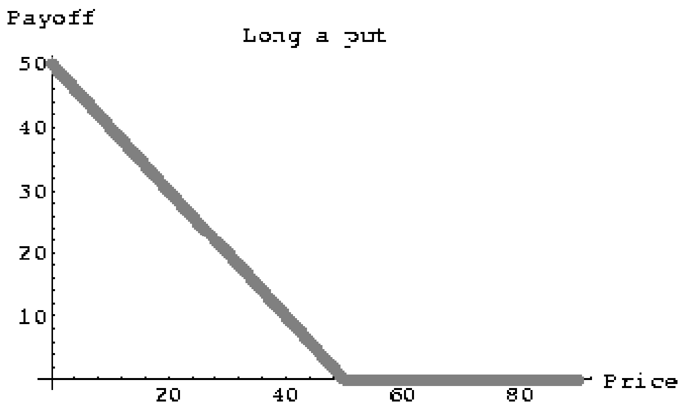
- 買權(call)給予持有人以X價格購買標的物權利
  - 到期日損益:max(S(T)-X,0)
  - 在到期日時,S(T)>X,持有人履約(exercise)
    - payoff=S(T)-X
  - 在到期日時,S(T)<X,持有人放棄合約

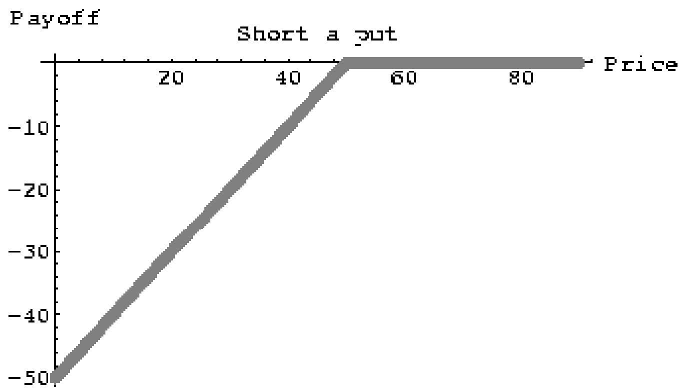
- 賣權(put)給予持有人以X價格售出標的物權利
  - 到期日損益:max(X-S(T),0)
- 購買選擇權的成本稱為權利金

Payoff
（假設無權利金，歐式選擇權）









Payoffs for European Options on Maturity

- Suppose that you have bought one European put and an European call on DELL with the same strike price of $55. The payoffs of your options certainly depend on the price of DELL on maturity

Stock Price	$30	40	50	60	70	80
Call Value	0	0	0	5	15	25
Put Value	25	15	5	0	0	0

選擇權的種類

- 歐式選擇權(European option)只能在到期日時才能決定是否履約
  - 前一頁的損益針對歐式選擇權
- 美式選擇權(American option)可在到期日之前履約
  - exercise at time t:
    - Call: S(t)-X
    - Put: X-S(t)
- 美式選擇權的權利金>=歐式選擇權

選擇權的價值

- 對選擇權買方而言，為了取得未來買進(賣出)的權利，自然必須付出代價，此代價便是選擇權的價值，也就是權利金(Premium)。
- 權利金和一般現貨市場的報價一樣，隨著買方與賣方願意支付與接受的情況，形成市場上的供需，當價格達到買賣雙方均能接受的條件時便可成交。
- 選擇權之權利金是由內含價值(Intrinsic value)與時間價值(time value)所組成
  - 內含價值即是選擇權履約價格與現貨價格之差，時間價值則是權利金扣除內含價值的部分。

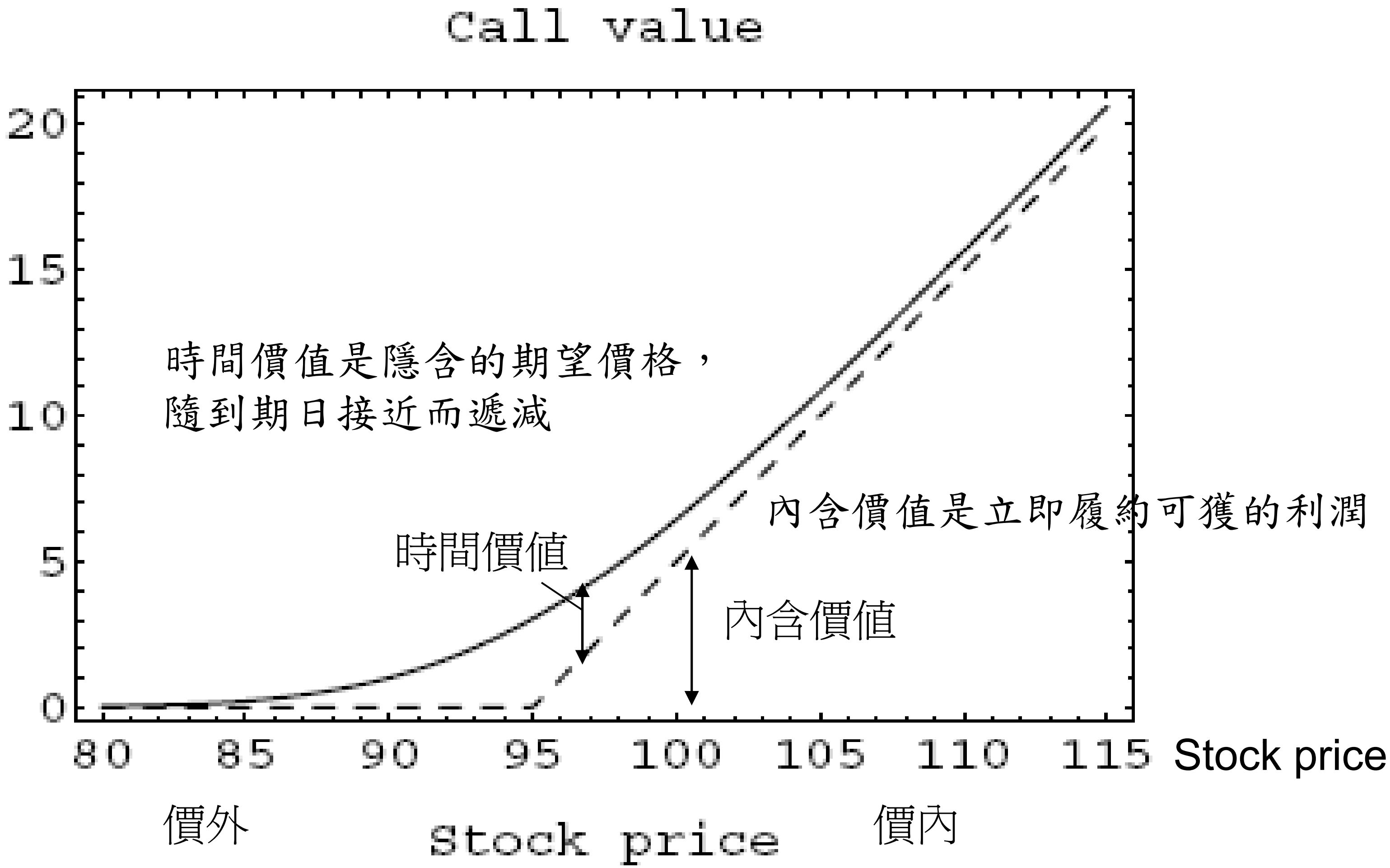
權利金(Premium) vs. 保證金(Margin)

- 權利金為選擇權之價值，但與保證金不同。
- 賣方賣出選擇權之後，背負履約的義務，為保證到期能履行義務，故要求賣方繳存一定金額之保證金。
- 保證金繳交之對象：買權、賣權的賣方。
- 需進行每日結算，以控制違約風險。

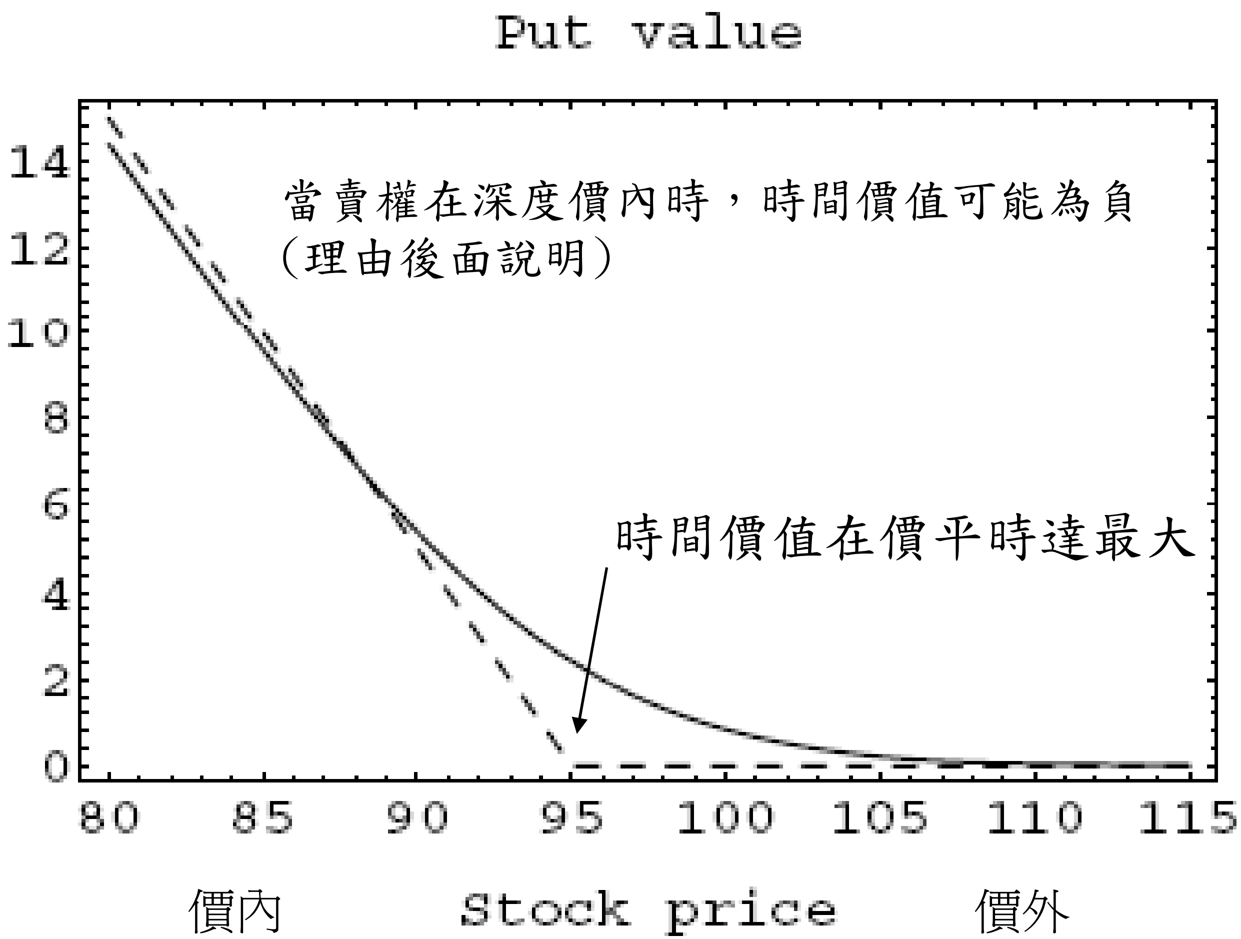
時間價值

- 影響時間價值的二個重要因素：到期日、標的物的價格波動率(下一章會詳細探討)。
- 接近到期日，時間價值遞減的速度愈快；在到期日時，時間價值降為零，只剩下內在價值的部分。
- 當時間價值減少時，獲利的是選擇權的賣方，因此在到期日接近時，反而對賣方有利。

Call Value
(Intrinsic value + time value)



Put Value
(Intrinsic value + time value)



Some Terminologies

- In the money
  - Call : $S>X$, Put : $S<X$
- At the money
  - Call, put : $S=X$
- Out of the money
  - Call : $S<X$, Put : $S>X$
- In the money options at expiration should be exercised.

影響選擇權價格的因素

	標的物價格	履約價格	無風險利率	到期期限	標的物價格波動率
Call	＋	－	＋	＋	＋
Put	－	＋	－	不一定	＋

（各因素影響的原因後面會詳細介紹）

波動性愈大的現貨，其選擇權的價格愈高。以向上波動而言，買權獲利無限而賣權損失有限；以向下波動來說，買權損失有限而賣權最大獲利為履約價格

Theory of Rational Option Pricing

- 以下將介紹幾個關於選擇權價格的重要定理，在這之前必須先了解兩個重要概念。
- No arbitrage and Dominance Principle
- Put-Call Parity

Notation

- S: Current stock price
- X: Strike price of option
- t: Time to expiration of option (unit: year)
- r: Continuously compound a year risk-free rate of interest
- C: value of American call option to buy one share
- c: value of European call option to buy one share
- P: value of American put option to buy one share
- p: value of European put option to buy one share

No arbitrage concept

- If two securities have the exactly the same payoff or cash flows in every state of each future period, these two securities should have the same price; otherwise there is an arbitrage opportunity.

Dominance Principle

- A risk-less arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances.
- The portfolio dominance principle says portfolio A should be more valuable than B if A's payoff is at least as good under all circumstances and better under some.

Put-Call Parity

- Let p(X,t) and c(X,t) be the prices of a European put and a call with same strike prices of X and maturity of t. Then we have

$$c(X,t) = S_0 + p(X,t) - Xe^{-rt}$$

Or

$$c(X,t) + Xe^{-rt} = S_0 + p(X,t)$$

若已知買權價格，透過賣買權平價理論可以推的賣權價格，不過有幾個重要前提必須成立：歐式選擇權、買賣權的履約價格與到期日均相同

Put-Call Parity (Proof)

$$c + PV(X) = p + S$$

在連續複利的假定下$PV(X)=Xe^{-rt}$

- Consider the following two portfolios.
- A: Buy one European call option plus an zero coupon bond amount of cash equal to PV(X) X is par value
- B: Buy one European put option plus one share stock.

	Initial cost	$S_t \geqq X$	$S_t < X$
A	c+PV(X)	$(S_t - X)+X= S_t$	X
B	p+S	S_t	$S_t +(X- S_t)=X$

未來的報酬均相同，在無套利的空間下，期初的投資成本應該要相同。

Consequences of Put-Call Parity

- There is only one kind of European option because the other can be replicated from it in combination with the underlying stock and risk-less lending or borrowing.

(1) $S = c - p + PV(X)$ says a stock is equivalent to a portfolio containing a long call, a short put, and lending PV(X).

(2) $c - p = S - PV(X)$ implies a long call and a short put amount to a long position in stock and borrowing the PV of the strike price .

選擇權價格關係

- 選擇權的價格必須滿足特定的關係
  - 否則存在套利的空間
  - 例如:選擇權的價格>=0
    - Otherwise:
      - Long a option (Gain initial benefits.)
      - Exercise the option if it is beneficial.

Maturity and Option Value

- ***Theorem1***

  An American call (put) with a longer time to expiration cannot be worth less than an otherwise identical call (put) with a shorter time to expiration

Proof

- Suppose instead that $C_{t1} > C_{t2}$ where $t_1 < t_2$.
- Buy C_{t2} and sell C_{t1} to generate a net cash flow of $C_{t1} - C_{t2}$ at time zero.
- Exercise C_{t2} when C_{t1} is exercised.
- Otherwise, sell C_{t2} at time t1.

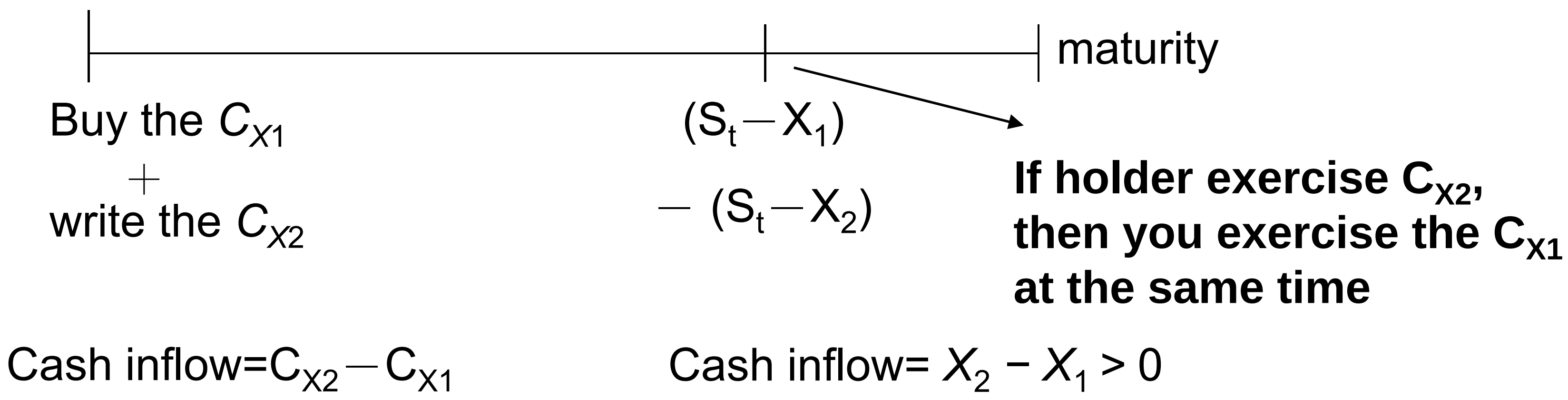
Strike Price and Option Value

- ***Theorem2***

  A call (put) option with a higher (lower) strike price cannot be worth more than an otherwise identical call (put) with a lower (higher) strike price.

Proof

- Let the two strike prices be $X_1 < X_2$. Suppose $C_{X1} < C_{X2}$ instead.



台股指數選擇權

由此圖可看出履約價格與買(賣)權的關係

月份: 2007/04　當天大盤收盤指數爲7757

買權							賣權						
履約價	時間	成交價	買價	賣價	漲跌	總量	履約價	時間	成交價	買價	賣價	漲跌	總量
6900	09:55	**820**	820	835	▽15	11	6900	13:43	**9.1**	8.6	9.1	▽2.9	1674
7000	11:27	**740**	690	740	△5	9	7000	13:44	**12**	12	12.5	▽5	959
7100	11:30	**630**	630	645	▽10	22	7100	13:44	**18**	17	18	▽8.5	2125
7200	13:36	**550**	540	550	△5	48	7200	13:44	**25**	26	27	▽12	5237
7300	13:37	**470**	445	470	△13	72	7300	13:44	**38.5**	38.5	39.5	▽11.5	6776
7400	13:40	**397**	372	397	△17	105	7400	13:44	**55**	55	56	▽14	6650
7500	13:44	**307**	304	307	△3	158	7500	13:44	**78**	78	79	▽21	7271
7600	13:44	**235**	231	235	▽2	335	7600	13:44	**109**	109	110	▽22	4500
7700	13:44	**173**	173	174	▽3	1816	7700	13:44	**148**	148	149	▽22	4786
7800	13:44	**120**	120	121	▽9	6088	7800	13:44	**194**	195	196	▽21	1043
7900	13:44	**79**	80	81	▽8	10637	7900	13:44	**258**	250	258	▽32	173
8000	13:44	**48.5**	48.5	49	▽6.5	13697	8000	13:28	**323**	323	332	▽33	175
8200	13:44	**15.5**	15.5	16.5	▽3	8519	8200	11:58	**493**	493	505	▽17	10
8400	13:41	**4.1**	4.1	4.2	▽0.6	1032	8400	—	—	—	—	—	—
8600	12:08	**1**	1	2	▽0.7	64	8600	08:47	**855**	825	855	▽15	7
8800	13:29	**0.6**	0.6	0.9	△0.1	27	8800	—	—	—	—	—	—

Upper bounds

- **Theorem3**

  A call is never worth more than the stock price, an American put is never worth more than the strike price, and a European put is never worth more than the PV of the strike price.

- If the call value exceeded the stock price, then we earn risk-less profit by longing stock & shorting call.
- If the put value exceeded the strike price, writing a cash-secured put earns arbitrage profits.

Intrinsic value and Call Value

- **Theorem4**

  A European call on a non-dividend-paying stock is never worth less than its intrinsic value, we prove

  $c \geqq$ Max (S-PV(X), 0) instead

  See later slide

- **Theorem5**

  A*n American call on a non-dividend-paying stock will never be exercise prior to expiration, and hence, it has the same value as a European call.*

  See later slide

Proof (Theorem4)

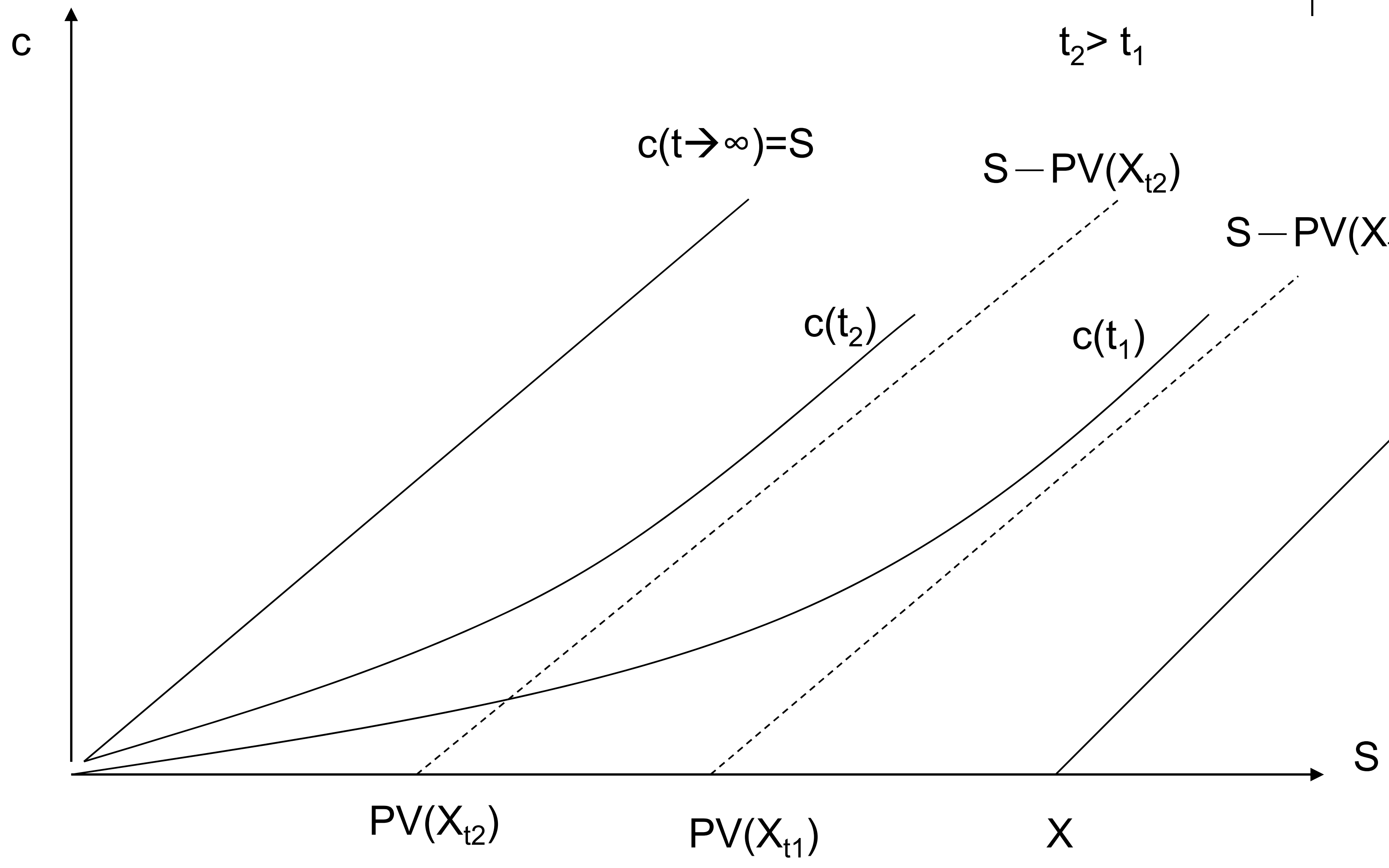
- Consider the following two investment:
- A: Buy the one European call for c   Total investment: =c+ PV(X)

  Buy one zero bond at price PV(X)
- B: Buy the common stock for S   Total investment: S
- Suppose at the end of t years, the common stock price is St.

	A	B	
St $\leqq$ X	X	S_t	A dominate B
St $>$ X	$(S_t-X)+X=S_t$	S_t	

By dominance principle, A will dominate B, c+ PV(X) $\geqq$ S . It implies that C $\geqq$ Max (S-PV(X), 0)

Call (different maturity)



Proof(Theorem5)

- From Theorem 4, $c \geqq$ Max (S-PV(X), 0), because the owner of an American call has all the exercise opportunities, we must have $C \geqq c$.

  → $C \geqq$ Max (S-PV(X), 0)

- Given $r > 0$, it follows that $S - PV(X) > S - X$.
- If it were optimal to exercise early, $C=S-X$. We deduce it can never be optimal to exercise early.

Special Example : Dividend Case (Early Exercise of American Calls)

- Surprisingly, an American call will only be exercised at expiration or just before an ex-dividend date.
- Early exercise may become optimal for American calls on a dividend-paying stock.
  - Stock price declines as the stock goes ex-dividend.
- This point will be discussed further in next chapter.

Intrinsic value and Put Value

- **Theorem6**

  For European puts, ***p ≥ Max (PV(X) − S, 0).***

- A European put on a non-dividend-paying stock may be worth less than its intrinsic value.
- Proof

$$\because c + PV(X) = p + S$$

$$\Rightarrow p = PV(X) - S + c \geq PV(X) - S$$

$$as\ the\ put\ goes\ deeper\ in\ the\ money, c \approx 0$$

$$\Rightarrow p \approx (X - S) + \big(PV(X) - X\big) < X - S$$

Early Exercise of American Puts

- **Theorem7**

It can be optimal to exercise an American put option on a non-dividend paying stock early. Besides, ***P ≥ Max (X − S, 0).***

$$\text{As the put goes deeper in the money}, c \approx 0$$

$$p = PV(X) - S + c \approx (X - S) + \left(PV(X) - X\right) < X - S$$

The value to exercise the option immediately.

$$P = \max(\text{continuation value}, X - S, 0) \geq \max(X - S, 0)$$

Summary (Option Bounds)

(If no dividends)	Upper bounds	lower bounds
European call	S	max(S − PV(X),0)
American call	S	max(S − PV(X),0)
European put	PV(X)	max(PV(X) − S,0)
American put	X	max(X − S,0)

Convexity of Option Prices

- Theorem8

If C and P is a rationally determined American call and put price, then C and P is convex function of its exercise price (X)

$$\mathbf{C_{X2} \le \omega C_{X1} + (1-\omega)\, C_{X3}}$$
$$\boldsymbol{P_{X2} \le \omega P_{X1} + (1-\omega)\, P_{X3}}$$

three otherwise identical calls with strike prices $X_1 < X_2 < X_3$

where $\omega \equiv (X_3 - X_2)/(X_3 - X_1)$.

Remarks: The above arguments can also be applied to European options.

Robert C. Merton (1973)

Proof (1)

Assume $C_{X2} > \omega C_{X1} + (1 - \omega) C_{X3}$

- Write C_{X2}, buy ωC_{X1}, and buy $(1 - \omega)C_{X3}$ to generate a positive cash flow now.
- If the short call is not exercised before expiration, hold the calls until expiration.

	$S \le X_1$	$X_1 < S \le X_2$	$X_2 < S < X_3$	$X_3 \le S$
$-C_{X_2}$	0	0	$X_2 - S$	$X_2 - S$
C_{X_1}	0	$\omega(S - X_1)$	$\omega(S - X_1)$	$\omega(S - X_1)$
C_{X_3}	0	0	0	$(1 - \omega)(S - X_3)$
Net	0	$\omega(S - X_1)$	$\omega(S - X_1) + (X_2 - S)$	0

Proof (2)

- Suppose the short call is exercised early when the stock price is S.
- If $\omega C_{X1} + (1 - \omega) C_{X3} > S - X_2$, sell the long calls to generate positive net cash flow.

  $\omega C_{X1} + (1 - \omega) C_{X3} - (S - X_2) > 0.$

- Otherwise, exercise the long calls and deliver the stock.
- The net cash flow is $-\omega X_1 - (1 - \omega) X_3 + X_2 = 0$.

there is an arbitrage profit!

Class Exercise

利用下圖試著驗算theorem8 (eg. K=6900,7000,7100)

月份: 2007/04

買權						
履約價	時間	成交價	買價	賣價	漲跌	總量
6900		**820**	820	825	▽15	11
7000		**740**	735	740	△5	9
7100		**630**	630	635	▽10	22
7200		**550**	540	550	△5	48
7300		**470**	445	470	△13	72
7400		**397**	372	397	△17	105
7500		**307**	304	307	△3	158
7600		**235**	231	235	▽2	335
7700		**173**	173	174	▽3	1816
7800		**120**	120	121	▽9	6088
7900		**79**	80	81	▽8	10637
8000		**48.5**	48.5	49	▽6.5	13697
8200		**15.5**	15.5	16.5	▽3	8519
8400		**4.1**	4.1	4.2	▽0.6	1032
8600		**1**	1	2	▽0.7	64
8800		**0.6**	0.6	0.9	△0.1	27

假設上列爲同一時間的報價

Portfolio Option

- **Theory9**

If k is a positive constant. Let C and C_Q *be two call options with the same underlying asset and maturity, If* $Q=kS$*;* $X_Q=kX$*, then we have* $C_Q=k \times C$

- **Theory10**

Consider a portfolio of non-dividend-paying assets with weights ωi. Let ci denote the price of a European call on asset i with strike price Xi. Then the call on the portfolio with a strike price $X \equiv \sum_i w_i X_i$ *has a value at most* $\sum_i w_i c_i$ *.All options expire on the same date.*

#Homework 9-1

- Prove Theorem 10.

Consider a portfolio of non-dividend-paying assets with weights ωi. Let ci denote the price of a European call on asset i with strike price Xi. Then the call on the portfolio with a strike price $X \equiv \sum_i w_i X_i$ *has a value at most .* $\sum_i w_i c_i$ *.All options expire on the same date.*

#Homework 9-2
The relationship between the future and the options prices

- Denote the prices for call and put options with strike price X and maturity T as Vc and Vp.
- Denote the price for the future matured at T as Vf
- Consider the following two strategies:
- 1: Long a call and short a put:
  - Initial payoff: Vp-Vc   At maturity: ST-X
- 2: Short a future: At maturity: Vf-ST
- Derive the relationship between Vp, Vc and Vf to avoid arbitrage

#Homework 10

- Write a program to check the quotes (simplified as follows) in the stocks, futures, and options markets for arbitrage opportunities. (r=1.844%, maturity=1 month

TAIEX	7757	7758	Future	7756	7760

月份: 2007/04

買權							賣權						
履約價	時間	成交價	買價	賣價	漲跌	總量	履約價	時間	成交價	買價	賣價	漲跌	總量
6900	09:55	**820**	820	835	▽15	11	6900	13:43	**9.1**	8.6	9.1	▽2.9	1674
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7200	13:36	**550**	540	550	△5	48	7200	13:44	**25**	26	27	▽12	5237
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7400	13:40	**397**	372	397	△17	105	7400	13:44	**55**	55	56	▽14	6650
7500	13:44	**307**	304	307	△3	158	7500	13:44	**78**	78	79	▽21	7271
7600	13:44	**235**	231	235	▽2	335	7600	13:44	**109**	109	110	▽22	4500
7700	13:44	**173**	173	174	▽3	1816	7700	13:44	**148**	148	149	▽22	4786
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7900	13:44	**79**	80	81	▽8	10637	7900	13:44	**258**	250	258	▽32	173
8000	13:44	**48.5**	48.5	49	▽6.5	13697	8000	13:28	**323**	323	332	▽33	175
8200	13:44	**15.5**	15.5	16.5	▽3	8519	8200	11:58	**493**	493	505	▽17	10
8400	13:41	**4.1**	4.1	4.2	▽0.6	1032	8400	—	—	—	—	—	—
8600	12:08	**1**	1	2	▽0.7	64	8600	08:47	**855**	825	855	▽15	7
8800	13:29	**0.6**	0.6	0.9	△0.1	27	8800	—	—	—	—	—	—

#Homework 10

- Remark:
  - Assume that we can long/short TAIEX
  - Each point for TAIEX, futures, and options worth 1 TWD.
  - All options are European ones
  - The transaction costs for every trading =1 except saving/borrowing
    - Ex: Long a call, short a put, sell a TAIEX, save $
      - Transaction cost =3
    - Transaction costs must be considered when performing arbitrage
  - Program inputs: The bid and ask prices for TAIEX, TAIEX futures, calls, puts.
  - Program outputs: Arbitrage strategies and profits.

選擇權的組合

- 選擇權和標的物可用不同形式組合(策略),組合出不同的損益
  - Hedge: Option +Underlying asset
  - Spread：同一類型的選擇權(只用call or put)
  - Combination：不同類型的選擇權。

Covered Position: Hedge

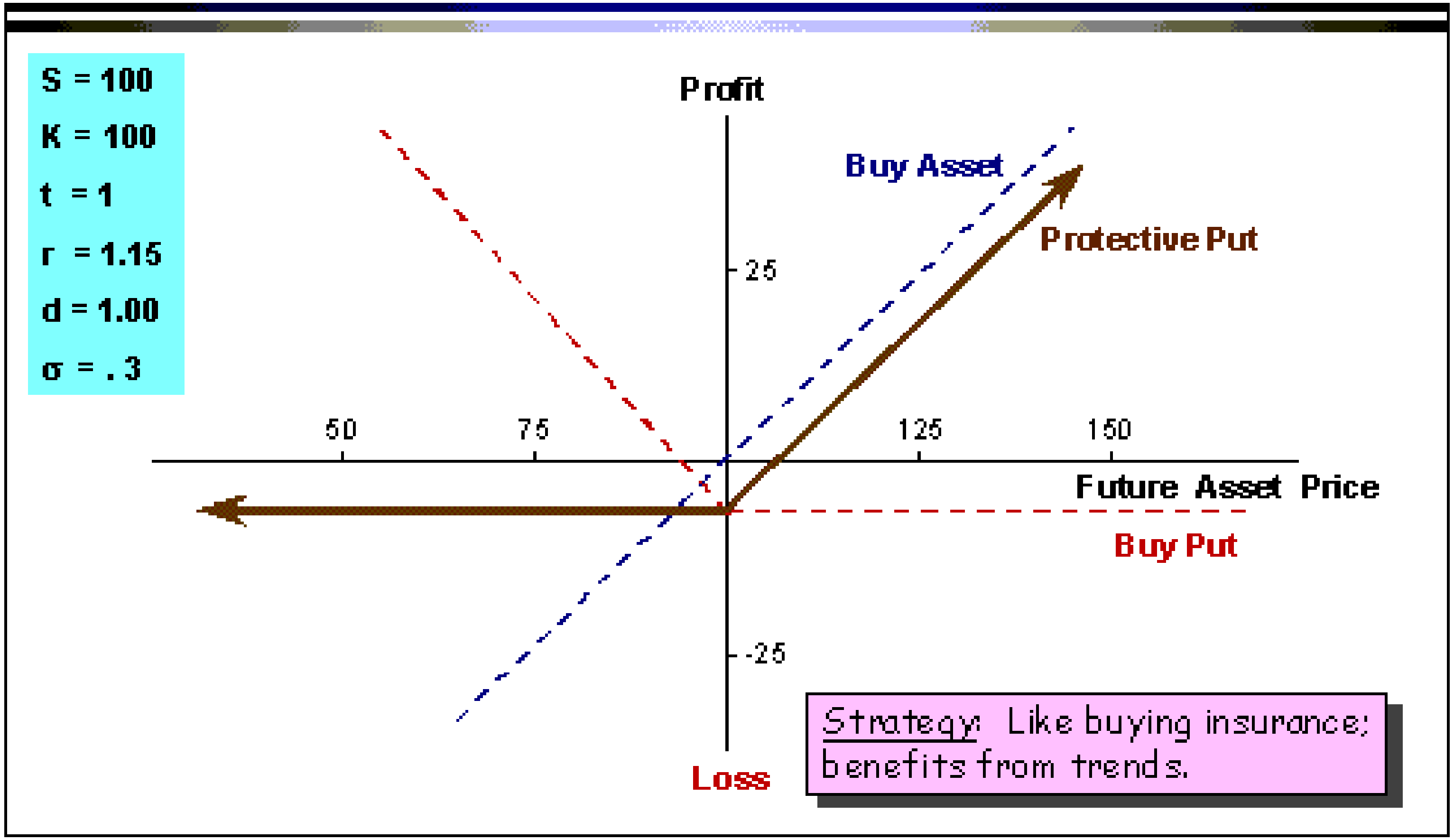
- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Protective put: A long position in stock with a long put.

(Example：買進股票同時買進該股票的賣權，在股價下跌時，賣權的報酬可以彌補股票的損失)

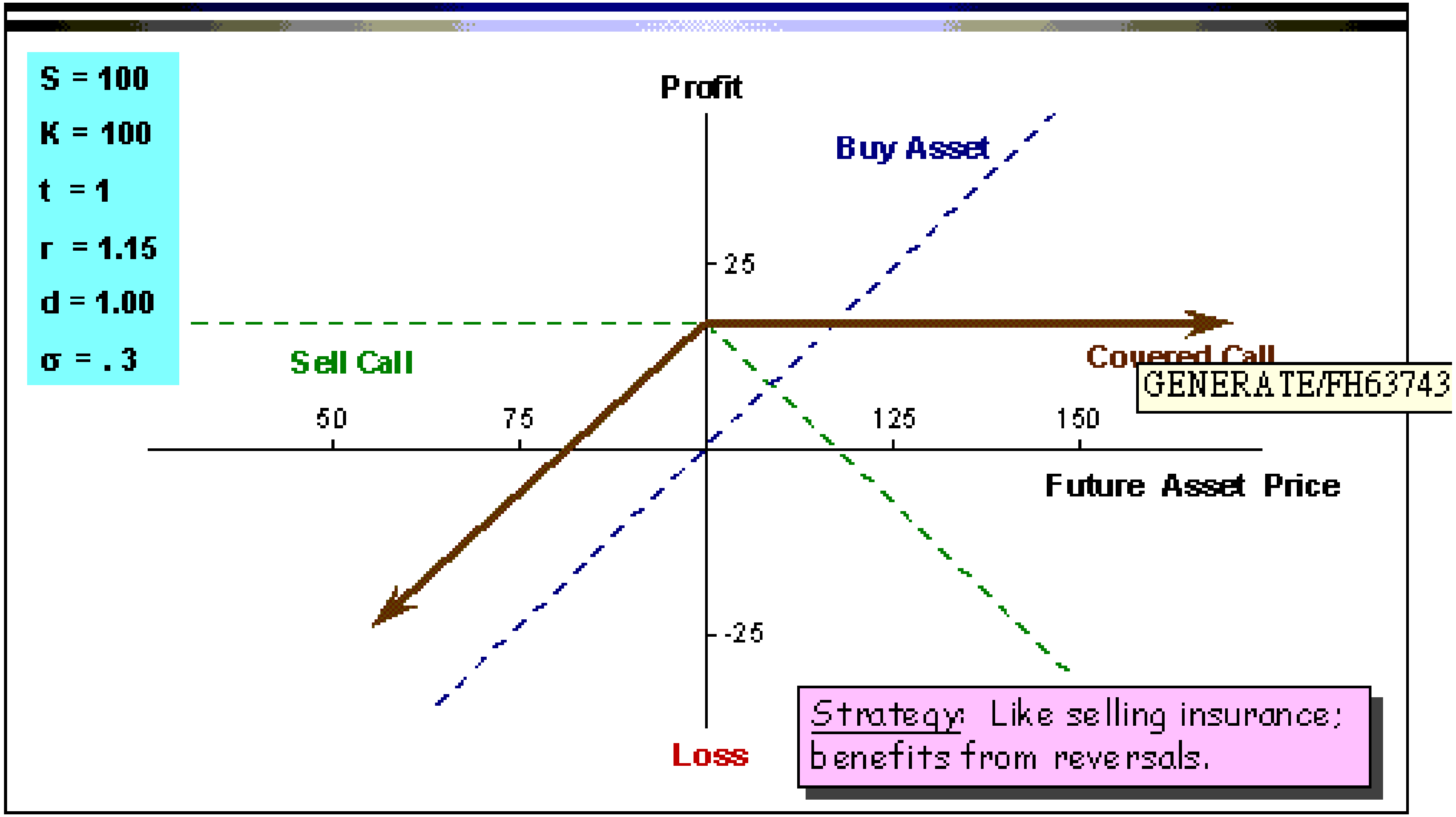
- Covered call: A long position in stock with a short call.

(Example：買進股票同時賣出該股票的買權，在股價上漲時，買權之虧損可由股票的收益彌補)

Protective Put



Covered Call



Example：Protective Put

- 阿貴持有台指ETF(假設台股指數為5200點)，並同時買一張履約價5200賣權，權利金210點(契約乘數為一點50元)，透過此策略可保護指數低於履約價所造成的損失。
- 期初支付210點權利金：10500
- 最大可能損失:210點
- 損益兩平點:5410(5200+210)

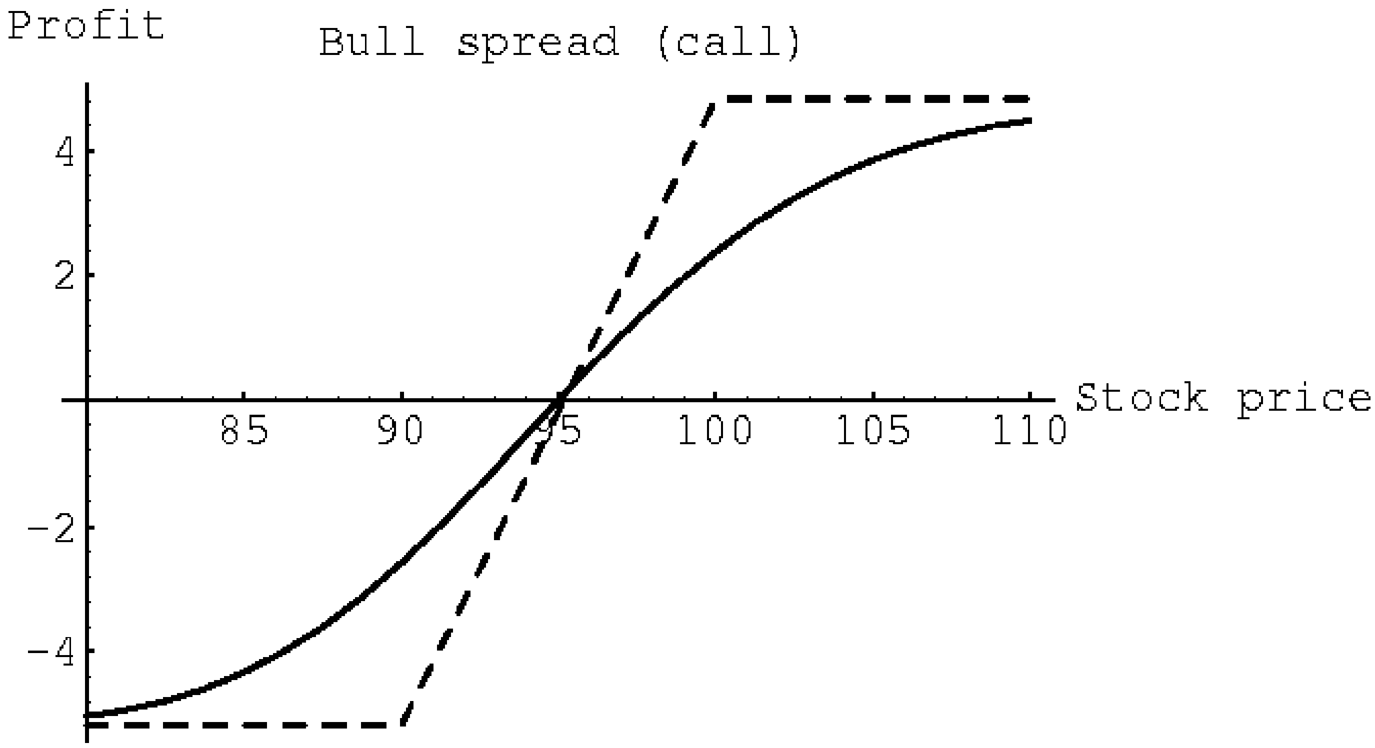
可動手畫出損益圖!

到期指數	ETF損益	賣權損益	權利金	淨損益(點)
5600	400	0	-210	190
5400	200	0	-210	-10
5200	0	0	-210	-210

Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with $X_L < X_M < X_H$.
- Example：A bull call spread consists of ***a*** long X_L call and ***a*** short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
  - The maximum loss is $C_L - C_H$.

Bull Spread



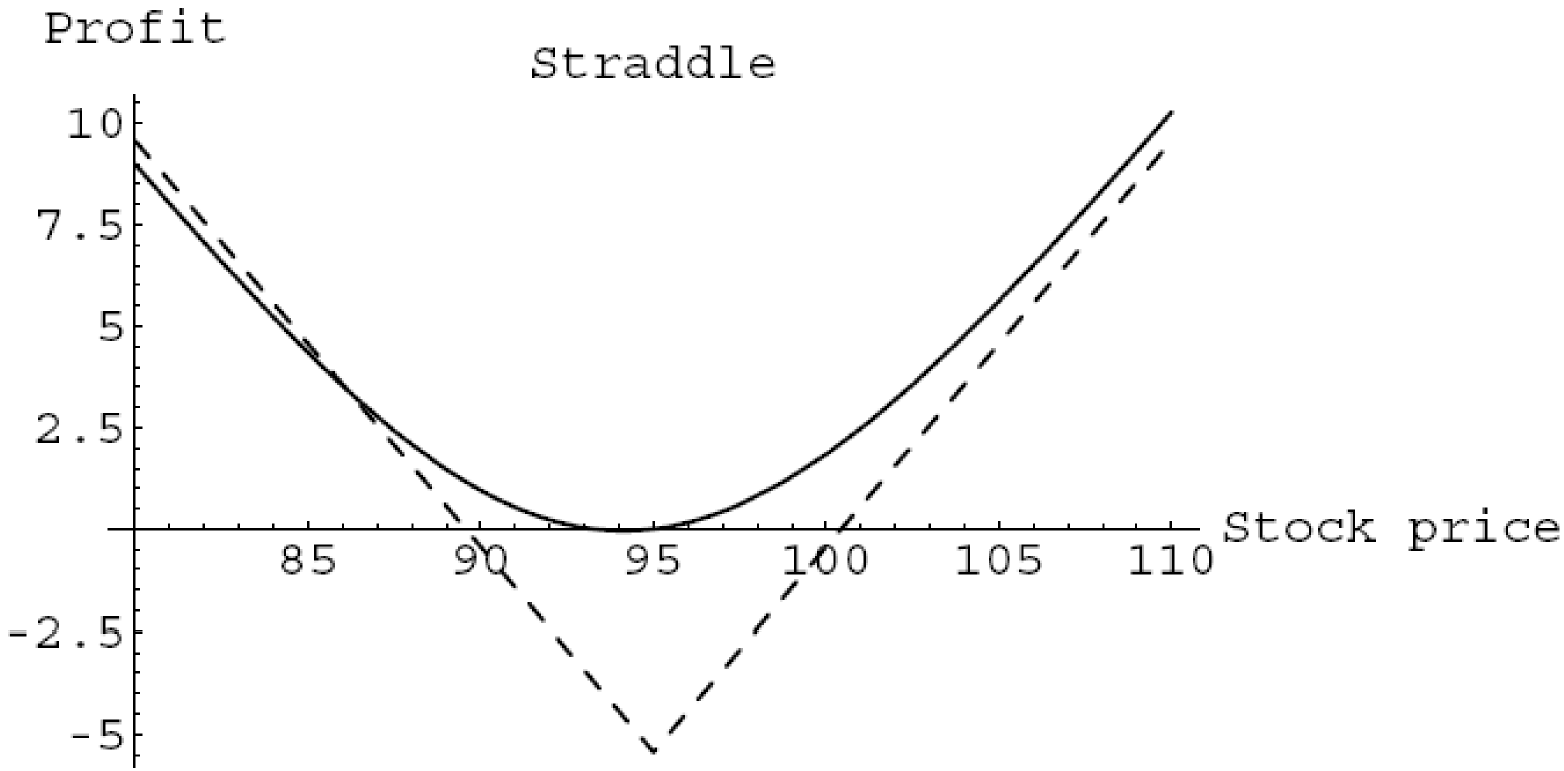
In Class Exercise (Bear Spread)

- Construct a bear spread with two put options.
  - Long a put (strike price=X_H)
  - short a put (strike price=X_L)
- Draw the payoff chart for the bear spread
- Calculate the maximum profits/loss.

Covered Position: Combination

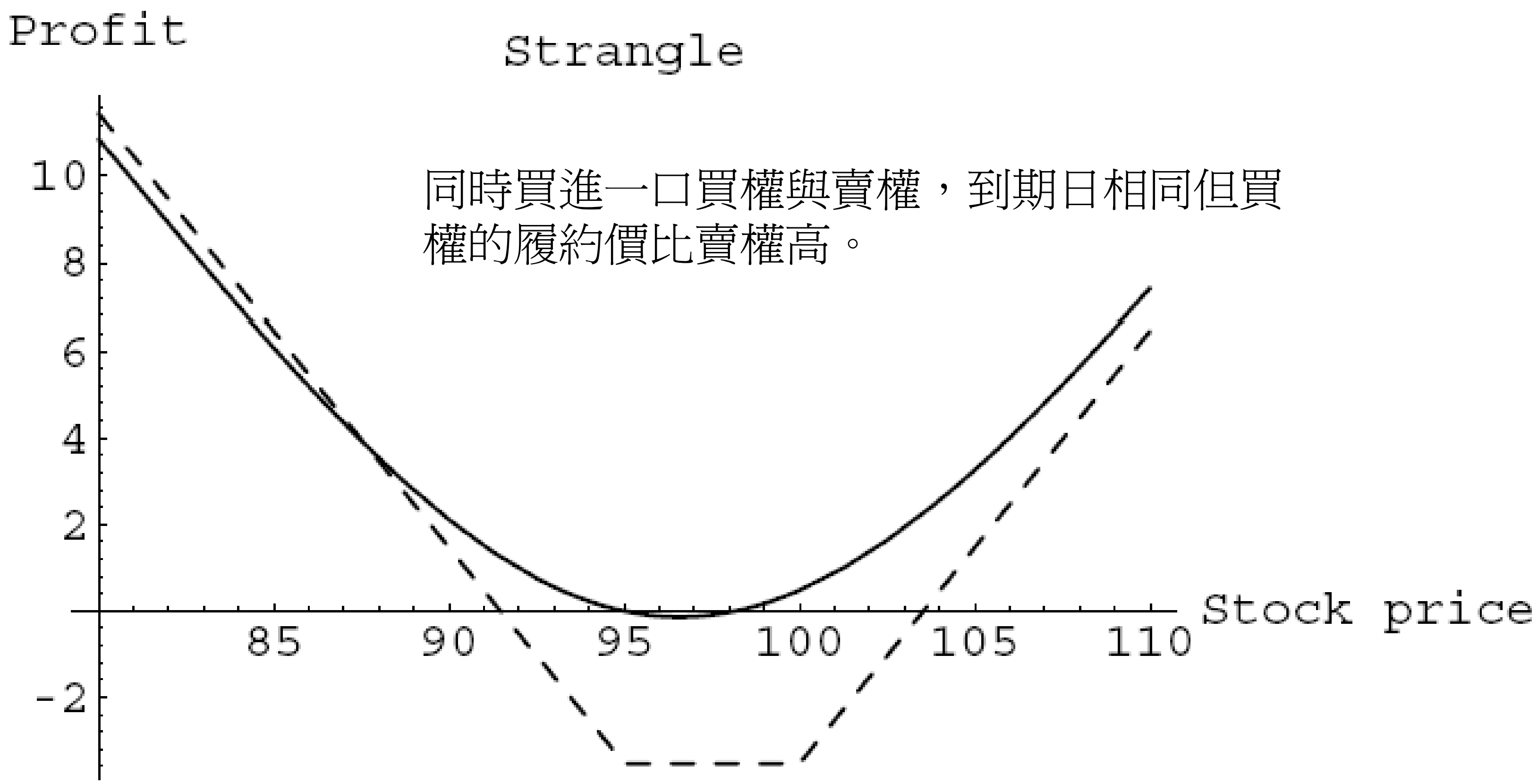
- A combination consists of options of different types on the same underlying asset, and they are either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.

Straddle



同時買進一口買權與賣權，履約價與到期日均相同。

Strangle



同時買進一口買權與賣權，到期日相同但買權的履約價比賣權高。

Example (Straddle)

- Say the Federal Reserve has indicated that it is strongly considering raising the Fed Funds rate to control inflation. A shape increase in interest rates may send stocks sharply lower, while a decrease in interest rate may boost XYZ to an all-time high.
- An investor expects that either of these outcomes could move the market up or down by 5% or more over a timeframe of approximately one month.

Example (Straddle)

- Index XYZ is currently at 100. The investor buys a one-month XYZ 100 call for $1.70, and a one-month XYZ 100 put for $1.50.
- The cost for the straddle is: $1.70 (call) + $1.50 (put) = $3.20.
- The total premium paid is therefore: $3.20 x 100 multiplier = $320.

Example (Straddle)

- By purchasing the straddle the investor is saying that by expiration he anticipates index XYZ to have either risen above the upside break-even point or below the downside break-even point:
- **Upside Break-Even Point:** $100 + \$3.20 = 103.20$
  Downside Break-Even Point: $100 - \$3.20 = 96.80$

If you are interested in any strategies, you may access to the website http://www.cboe.com/Strategies